THE PROBABILITY DISTRIBUTIONS OF THE FIRST HITTING TIMES OF BESSEL PROCESSES

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Abstract. We consider the first hitting times of the Bessel processes. We give explicit expressions for the distribution functions and for the densities by means of the zeros of the Bessel functions. The results extend the classical ones and cover all the cases.

1. Introduction.

In this article we consider the first hitting time of the Bessel process, which itself is an interesting object and is one of the important tools to study several problems in probability theory. By general theory of one-dimensional diffusion processes, the Laplace transform of the distribution satisfies an eigenvalue problem for the generator and it is given by a ratio of the modified Bessel functions.

Except some special cases it is not easy to invert the Laplace transforms. When the index ν of the Bessel process is a half integer $n+1/2$, $n \in \mathbb{N}$, the Macdonald function K_ν is of a simple form. In this case, it turns that $K_{\nu+1}/K_\nu$ is represented by the ratio of polynomials. With the help of the partial fraction decomposition, Hamana [8, 9] recently has inverted the Laplace transform and applied the results to show the explicit form and the asymptotic behavior of the expected volume of the Wiener sausage for the odd dimensional Brownian motion. The method used in [9] requires some formulae for the zeros of K_ν.

The purpose of this paper is to show explicit formulae for the distribution functions and the densities by means of the zeros of the Bessel functions. The results extend the classical ones, for example, due to Ciesielski and Taylor [4], and cover all the cases.

In order to prove the results we represent the ratio of the modified Bessel functions by using contour integrals of functions easier to treat, and invert the Laplace transform. Recently Byczkowski et al [2, 3] have given similar but different expressions for the densities of the first hitting times, and applied the results to some study on geometric Brownian motions and hyperbolic Brownian motions. We use the same curve for the contour integral. But, we show a decomposition of the

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Bessel function ratio and use it, which makes our expression simpler. Moreover, the relation to the classical results in some special cases is clear.

This article is organized as follows. In the next Section 2 we give the main results. After showing two estimates for some ratios of the Bessel functions in Section 3, we prove the main results, Theorems 2.1 and 2.2, in Sections 4 and 5, respectively. Section 6 is devoted to the asymptotic behavior of the tail probability of the first hitting time, which is obtained as an application of the result. In the final Section 7 we show an addition formula for some ratio of the Bessel function, which is guessed from one of our results and is of independent interest.

2. The first hitting time of the Bessel processes.

For $\nu \in \mathbb{R}$ the one-dimensional diffusion process with infinitesimal generator

$$\mathcal{G}^{(\nu)} = \frac{1}{2}\frac{d^2}{dx^2} + \frac{2\nu+1}{2x}\frac{d}{dx} = \frac{1}{2x^{2\nu+1}}\frac{d}{dx}\left(x^{2\nu+1}\frac{d}{dx}\right)$$

is called the Bessel process with index ν. If $2\nu+2$ is a positive integer, the Bessel process is identical in law with the radial motion of a $(2\nu+2)$-dimensional Brownian motion. Hence, $2\nu+2$ is called the dimension of the Bessel process.

The classification of boundary points gives the following information. The end-point ∞ is a natural boundary for any $\nu \in \mathbb{R}$. For $\nu \geqq 0$, 0 is an entrance and not exit boundary. For $-1 < \nu < 0$, 0 is a regular boundary, which is instantly reflecting. For $\nu \leqq -1$, 0 is an exit but not entrance boundary. For more details, see [10, 14] for example.

For $a, b \geqq 0$ we denote by $\tau_{a,b}^{(\nu)}$ the first hitting time to b of the Bessel process with index ν starting at a. By general theory of one-dimensional diffusion processes, we can evaluate the Laplace transform of the distribution of $\tau_{a,b}^{(\nu)}$ by solving an eigenvalue problem. In fact, the function

$$x \mapsto E[e^{-\lambda \tau_{x,b}^{(\nu)}}]$$

is increasing (decreasing) on $[0,b)$ (resp. (b,∞)) and satisfies

$$\mathcal{G}^{(\nu)}u = \lambda u, \quad u(b) = 1.$$

The following expressions for $E[e^{-\lambda\tau_{a,b}^{(\nu)}}]$ is well known (cf. [6, 11]): for $\lambda > 0$, if $b > 0$ and $\nu > -1$,

$$E[e^{-\lambda\tau_{0,b}^{(\nu)}}] = \frac{(b\sqrt{2\lambda})^{\nu}}{2^{\nu}\Gamma(\nu+1)}\frac{1}{I_\nu(b\sqrt{2\lambda})}; \tag{2.1}$$

if $0 < a \leqq b$ and $\nu > -1$,

$$E[e^{-\lambda\tau_{a,b}^{(\nu)}}] = \frac{a^{-\nu}I_\nu(a\sqrt{2\lambda})}{b^{-\nu}I_\nu(b\sqrt{2\lambda})}; \tag{2.2}$$

if $0 < a \leqq b$ and $\nu \leqq -1$,

$$E[e^{-\lambda\tau_{a,b}^{(\nu)}}] = \frac{a^{-\nu}I_{-\nu}(a\sqrt{2\lambda})}{b^{-\nu}I_{-\nu}(b\sqrt{2\lambda})}; \tag{2.3}$$

if $a > 0$ and $\nu < 0$,

$$E[e^{-\lambda\tau^{(\nu)}_{a,0}}] = \frac{2^{\nu+1}}{\Gamma(|\nu|)(a\sqrt{2\lambda})^{\nu}} K_\nu(a\sqrt{2\lambda}); \tag{2.4}$$

if $0 < b \leqq a$ and $\nu \in \mathbb{R}$,

$$E[e^{-\lambda\tau^{(\nu)}_{a,b}}] = \frac{a^{-\nu}K_\nu(a\sqrt{2\lambda})}{b^{-\nu}K_\nu(b\sqrt{2\lambda})}. \tag{2.5}$$

Here Γ is the gamma function and I_ν and K_ν denote modified Bessel functions of the first and the second kinds of order ν, respectively. Both I_ν and K_ν are the solutions of the modified Bessel differential equation

$$z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} - (z^2+\nu^2)w = 0. \tag{2.6}$$

When $b > 0$ and $2\nu+2$ is a positive integer, Ciesielski and Taylor [4] have already shown that, for $t > 0$

$$P(\tau^{(\nu)}_{0,b} \leqq t) = 1 - \frac{1}{2^{\nu-1}\Gamma(\nu+1)}\sum_{k=1}^{\infty}\frac{j_{\nu,k}^{\nu-1}}{J_{\nu+1}(j_{\nu,k})}e^{-\frac{j_{\nu,k}^2}{2b^2}t},$$

where J_μ is the Bessel function of the first kind of order μ and $\{j_{\nu,k}\}_{k=1}^{\infty}$ is the increasing sequence of positive zeros of J_ν. Hence the density $\rho^{(\nu)}_{0,b}(t)$ of $\tau^{(\nu)}_{0,b}$ is given by

$$\rho^{(\nu)}_{0,b}(t) = \frac{1}{2^{\nu}\Gamma(\nu+1)b^2}\sum_{k=1}^{\infty}\frac{j_{\nu,k}^{\nu+1}}{J_{\nu+1}(j_{\nu,k})}e^{-\frac{j_{\nu,k}^2}{2b^2}t}. \tag{2.7}$$

We will use the notation $\rho^{(\nu)}_{a,b}$ to denote the probability density function of $\tau^{(\nu)}_{a,b}$. While we need to calculate the inverse Laplace transform of the right hand side of (2.1) when $2\nu+2$ is not a positive integer, it is possible to obtain it by the same methods as those they used to prove Theorem 1 in [4]. Thus we see that (2.7) is also valid when $b > 0$ and $\nu > -1$.

In the case $a > 0$ and $\nu < 0$, we can easily check

$$\rho^{(\nu)}_{a,0}(t) = \frac{2^{\nu}}{\Gamma(|\nu|)a^2}t^{\nu-1}e^{-\frac{a^2}{2t}},$$

by (2.4) and the formula

$$K_\nu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^{\nu}\int_0^\infty e^{-t-\frac{z^2}{4t}}t^{-\nu-1}dt.$$

Formulae (2.2) and (2.5) are found in [1], p.398, in the case of $\nu > 0$. Moreover, when $0 < a \leqq b$, the following formula is provided:

$$\rho^{(\nu)}_{a,b}(t) = \frac{b^{\nu-2}}{a^{\nu}}\sum_{k=1}^{\infty}\frac{j_{\nu,k}J_\nu(aj_{\nu,k}/b)}{J_{\nu+1}(j_{\nu,k})}e^{-\frac{j_{\nu,k}^2}{2b^2}t}.$$

We shall give a proof under more general situation.

Theorem 2.1. *Let $0 < a < b$. Then, for $\nu > -1$,*

$$P(\tau_{a,b}^{(\nu)} \leqq t) = 1 - 2\left(\frac{b}{a}\right)^{\nu} \sum_{k=1}^{\infty} \frac{J_\nu(aj_{\nu,k}/b)}{j_{\nu,k}J_{\nu+1}(j_{\nu,k})} e^{-\frac{j_{\nu,k}^2}{2b^2}t} \tag{2.8}$$

and, for $\nu \leqq -1$,

$$P(\tau_{a,b}^{(\nu)} \leqq t) = \left(\frac{b}{a}\right)^{2\nu} - 2\left(\frac{b}{a}\right)^{\nu} \sum_{k=1}^{\infty} \frac{J_{-\nu}(aj_{-\nu,k}/b)}{j_{-\nu,k}J_{-\nu+1}(j_{-\nu,k})} e^{-\frac{j_{-\nu,k}^2}{2b^2}t}. \tag{2.9}$$

It should be noted that the result by Ciesielski and Taylor [4] may be obtained by letting $a \to 0$ in (2.8) and using the asymptotic behavior of $J_\nu(z)$ as $z \to 0$. This theorem immediately shows that, if $0 < a < b$ and $\nu > -1$,

$$P(\tau_{a,b}^{(\nu)} > t) = 2\left(\frac{b}{a}\right)^{\nu} \frac{J_\nu(aj_{\nu,1}/b)}{j_{\nu,1}J_{\nu+1}(j_{\nu,1})} e^{-\frac{j_{\nu,1}^2}{2b^2}t}\{1 + o(1)\}.$$

Similar asymptotic result in the case where $a = 0$ and $2\nu + 2$ is an integer greater than 2 (Brownian case) was used in [4] to show the law of iterated logarithm for the total time spent by the Bessel process in $(0, b)$ as $b \downarrow 0$.

To give our result for the distribution functions of $\tau_{a,b}^{(\nu)}$ in the case of $0 < b < a$, we need to recall some facts about the zeros of the Bessel function K_ν. For $\nu \in \mathbb{R}$ we denote by $N(\nu)$ the number of zeros of K_ν. It is known that $N(\nu) = |\nu| - 1/2$ if $\nu - 1/2$ is an integer and that $N(\nu)$ is the even number closest to $|\nu| - 1/2$ otherwise. We remark that $N(\nu) = 0$ if $|\nu| < 3/2$ and $N(\nu) \geqq 1$ if $|\nu| \geqq 3/2$. Each zero, if exists, lies in the half plain $\{z \in \mathbb{C}\,;\, \mathrm{Re}(z) < 0\}$, denoted by $\mathbb{C}^-$. In this case, we write $z_{\nu,1}, z_{\nu,2}, \ldots, z_{\nu,N(\nu)}$ for the zeros. Since K_ν is a solution of (2.6), all zeros of K_ν are of multiplicity one by the uniqueness of the solution of ordinary differential equations. This means that all zeros of K_ν are distinct. If $\nu - 1/2$ is not an integer, there are no real zeros. For details, see [15], pp.511–513.

Theorem 2.2. *Let $0 < b < a$. For $\mu \geqq 0$ and $c > 1$, we set*

$$L_{\mu,c}(x) = \frac{\cos(\pi\mu)\{I_\mu(cx)K_\mu(x) - I_\mu(x)K_\mu(cx)\}}{\{K_\mu(x)\}^2 + \pi^2\{I_\mu(x)\}^2 + 2\pi\sin(\pi\mu)K_\mu(x)I_\mu(x)}.$$

(1) *If $\nu = \pm 1/2$,*

$$P(\tau_{a,b}^{(\nu)} \leqq t) = \left(\frac{b}{a}\right)^{\nu+|\nu|} \int_0^t \frac{a-b}{\sqrt{2\pi s^3}} e^{-\frac{(a-b)^2}{2s}} ds.$$

(2) *If $|\nu| < 3/2$ and $\nu \neq \pm 1/2$,*

$$\begin{aligned} P(\tau_{a,b}^{(\nu)} \leqq t) =& \left(\frac{b}{a}\right)^{\nu+|\nu|} \int_0^t \frac{a-b}{\sqrt{2\pi s^3}} e^{-\frac{(a-b)^2}{2s}} ds \\ &- \left(\frac{b}{a}\right)^{\nu} \int_0^t \frac{a-b}{\sqrt{2\pi s^3}} e^{-\frac{(a-b)^2}{2s}} \left[\int_0^\infty \frac{L_{|\nu|,a/b}(x)}{x} e^{-\frac{x(a-b)\sqrt{t}}{b\sqrt{s}}} dx\right] ds. \end{aligned}$$

(3) *If* $\nu - 1/2$ *is an integer and* $\nu \neq \pm 1/2$,

$$
\begin{aligned}
P(\tau_{a,b}^{(\nu)} \leqq t) = & \left(\frac{b}{a}\right)^{\nu+|\nu|} \int_0^t \frac{a-b}{\sqrt{2\pi s^3}} e^{-\frac{(a-b)^2}{2s}} ds \\
& - \left(\frac{b}{a}\right)^{\nu} \sum_{j=1}^{N(\nu)} \frac{K_\nu(az_{\nu,j}/b)}{z_{\nu,j} K_{\nu+1}(z_{\nu,j})} \int_0^t \frac{a-b}{\sqrt{2\pi s^3}} e^{-\frac{(a-b)^2}{2s} + \frac{z_{\nu,j}(a-b)\sqrt{t}}{b\sqrt{s}}} ds.
\end{aligned}
$$

(4) *If* $\nu - 1/2$ *is not an integer and* $|\nu| > 3/2$,

$$
\begin{aligned}
P(\tau_{a,b}^{(\nu)} \leqq t) = & \left(\frac{b}{a}\right)^{\nu+|\nu|} \int_0^t \frac{a-b}{\sqrt{2\pi s^3}} e^{-\frac{(a-b)^2}{2s}} ds \\
& - \left(\frac{b}{a}\right)^{\nu} \sum_{j=1}^{N(\nu)} \frac{K_\nu(az_{\nu,j}/b)}{z_{\nu,j} K_{\nu+1}(z_{\nu,j})} \int_0^t \frac{a-b}{\sqrt{2\pi s^3}} e^{-\frac{(a-b)^2}{2s} + \frac{z_{\nu,j}(a-b)\sqrt{t}}{b\sqrt{s}}} ds \\
& - \left(\frac{b}{a}\right)^{\nu} \int_0^t \frac{a-b}{\sqrt{2\pi s^3}} e^{-\frac{(a-b)^2}{2s}} \left[\int_0^\infty \frac{L_{|\nu|,a/b}(x)}{x} e^{-\frac{x(a-b)\sqrt{t}}{b\sqrt{s}}} dx \right] ds.
\end{aligned}
$$

This theorem gives the asymptotic behavior of $P(\tau_{a,b}^{(\nu)} > t)$ for $0 < b < a$, which will be discussed in Section 6. It should be mentioned that, when $\nu = 0$, it is obtained from (2.5). In fact, we have

$$
\begin{aligned}
\int_0^\infty e^{-\lambda t} P(\tau_{a,b}^{(\nu)} > t) dt &= \frac{1}{\lambda} \frac{K_0(b\sqrt{2\lambda}) - K_0(a\sqrt{2\lambda})}{K_0(b\sqrt{2\lambda})} \\
&= \frac{2\log(a/b)}{\lambda \log(1/\lambda)} \{1 + o(1)\}
\end{aligned} \tag{2.10}
$$

as $\lambda \to 0$. The Tauberian theorem immediately yields

$$
P(\tau_{a,b}^{(\nu)} > t) = \frac{2\log(a/b)}{\log t} + o\left(\frac{1}{\log t}\right) \tag{2.11}
$$

as $t \to \infty$ (cf. [5], p.446). In order to derive the second equality of (2.10), we have applied the asymptotic behavior of $K_0(x)$ as $x \downarrow 0$ (cf. (5.14)). We can directly deduce (2.11) from Theorem 2.2 without the Tauberian theorem, however, the calculation is left to the reader.

In case of $\nu \neq 0$, we can not obtain the convenient formula like (2.10) which admits us to apply the Tauberian theorem in a straightforward way.

Remark 2.3. It is well known (see, e.g., [14], p.450) that the probability laws of the Bessel processes with different indices are absolutely continuous. Hence formula (2.9) may be deduced from (2.8) and the results in Theorem 2.2 in the case of $\nu < 0$ may be proven from those of $\nu > 0$. Although our proofs work in all the cases, this remark gives a good check for the results.

3. Some estimates for Bessel functions.

In this section we give two estimates concerning J_μ and I_μ, which we use in the proof of Theorem 2.1. Throughout this section, we assume that $\mu > -1$ and C_i's are positive constants independent of the variable.

We first recall some facts about the Bessel functions. Let D is the set of points $z \in \mathbb{C} \setminus \{0\}$ with $|\arg z| < \pi$ and $\{j_{\mu,k}\}_{k=1}^{\infty}$ denotes the increasing sequence of positive zeros of J_μ. It is well known that J_μ has no other zeros in D (cf. [12], p.127 and [15], pp.478–484). By virtue of the fact that

$$J_\mu(ze^{im\pi}) = e^{im\pi\mu} J_\mu(z) \tag{3.1}$$

for $m \in \mathbb{Z}$ and $z \in D$ (see [15], p.75), we see that each $-j_{\mu,k}$ is also a zeros of J_μ for $k \geqq 1$. Moreover, in [15], p.77, we find

$$I_\mu(z) = \begin{cases} e^{-i\pi\mu/2} J_\mu(ze^{i\pi/2}) & \text{if } -\pi < \arg z \leqq \pi/2, \\ e^{i3\pi\mu/2} J_\mu(ze^{-i3\pi/2}) & \text{if } \pi/2 < \arg z \leqq \pi. \end{cases} \tag{3.2}$$

From this formula we see that the zeros of I_μ with $-\pi < \arg z \leqq \pi$ are $j_{\mu,k}e^{\pm i\pi/2}$ for $k \geqq 1$.

The following estimates for the Bessel functions of the third kind of order μ, denoted by $H_\mu^{(1)}$ and $H_\mu^{(2)}$, are useful for our purpose. Let $\delta > 0$ be given. Then it is known that, for $-\pi + \delta \leqq \arg z \leqq 2\pi - \delta$,

$$H_\mu^{(1)}(z) = \sqrt{\frac{2}{\pi z}} e^{i(z-\pi\mu/2-\pi/4)} \{1 + E_1(z)\} \tag{3.3}$$

and that, $-2\pi + \delta \leqq \arg z \leqq \pi - \delta$,

$$H_\mu^{(2)}(z) = \sqrt{\frac{2}{\pi z}} e^{-i(z-\pi\mu/2-\pi/4)} \{1 + E_2(z)\} \tag{3.4}$$

where $|E_1(z)| \leqq C_1/|z|$ and $|E_2(z)| \leqq C_2/|z|$. These estimates can be shown for $\mu > -1/2$ by the integral representation of the Bessel functions of the third kind. Formulae (3.3) and (3.4) are also valid for $\mu \leqq -1/2$, since

$$H_{-\mu}^{(1)}(z) = e^{i\pi\mu} H_\mu^{(1)}(z), \quad H_{-\mu}^{(2)}(z) = e^{-i\pi\mu} H_\mu^{(2)}(z).$$

The details are given in [12], p.121 and [15], pp.197–199.

The first lemma concerns the exponential growth of the modified Bessel function $I_\mu(z)$ as $|z| \to \infty$.

Lemma 3.1. *Let c and η be real numbers with $0 < c < 1$ and $\eta \geqq 1$. Then there exist constants C_3, C_4 and $\kappa_1(c,\eta)$ such that, if $R > \kappa_1(c,\eta)$ and $J_\mu(R) \neq 0$,*

$$\left| \frac{I_\mu(cz)}{I_\mu(z)} \right| \leqq \frac{1}{\sqrt{c}} e^{-2(1-c)\eta} \frac{2 + C_3/R}{1 - e^{-2} - C_4/R}$$

holds for every z with $|z| = R$ and $|\arg z| \leqq \pi/2 - \mathrm{Arcsin}(\eta/R)$.

Proof. For simplicity we write δ for $\operatorname{Arcsin}(\eta/R)$. By definition we have

$$J_\mu(z) = \frac{1}{2}\{H_\mu^{(1)}(z) + H_\mu^{(2)}(z)\} \tag{3.5}$$

for $z \in D$ (cf. [15], p.74). We first consider the case where $0 \leqq \arg z \leqq \pi/2 - \delta$. It follows from (3.1), (3.2) and (3.5) that

$$\begin{aligned} I_\mu(z) &= e^{i\pi\mu/2} J_\mu(ze^{-i\pi/2}) \\ &= \frac{1}{2} e^{-i\pi\mu/2}\{H_\mu^{(1)}(ze^{i\pi/2}) + H_\mu^{(2)}(ze^{i\pi/2})\}. \end{aligned}$$

Since $-\pi/2 \leqq \arg(ze^{-i\pi/2}) \leqq -\delta$, we obtain by (3.3) and (3.4) that

$$\begin{aligned} \frac{1}{2} e^{i\pi\mu/2} H_\mu^{(1)}(ze^{-i\pi/2}) &= \frac{1}{\sqrt{2\pi z}} e^z \{1 + E_1(ze^{-i\pi/2})\}, \\ \frac{1}{2} e^{i\pi\mu/2} H_\mu^{(2)}(ze^{-i\pi/2}) &= \frac{1}{\sqrt{2\pi z}} e^{-z+i(\mu+1/2)\pi} \{1 + E_2(ze^{-i\pi/2})\}, \end{aligned}$$

which yields that

$$I_\mu(z) = \frac{e^z}{\sqrt{2\pi z}} \left[1 + E_1(ze^{-i\pi/2}) + e^{-2z+i(\mu+1/2)\pi}\{1 + E_2(ze^{-i\pi/2})\}\right].$$

Now let $z = Re^{i\theta}$ and $0 \leqq \theta \leqq \pi/2 - \delta$. Note that

$$\cos\theta \geqq \cos\left(\frac{1}{2}\pi - \delta\right) = \sin\delta = \frac{\eta}{R}.$$

Then it is easy to see that

$$|e^{-2z \pm i(\mu+1/2)\pi}| = e^{-2R\cos\theta} \leqq e^{-2\eta} \leqq e^{-2}. \tag{3.6}$$

Hence, if we write

$$I_\mu(z) = \frac{e^z}{\sqrt{2\pi z}} \left\{1 + e^{-2z+i(\mu+1/2)\pi} + E_4(z)\right\}, \tag{3.7}$$

then there exists a constant C_5 such that $|E_4(z)| \leqq C_5/|z|$.

Since $I_\mu(j_{\mu,k}e^{\pm i\pi/2}) = 0$ for $k \geqq 1$, we have that $I_\mu(z) \neq 0$ under the assumption of this lemma. Moreover, by virtue of (3.6), we obtain

$$\left|1 + e^{-2z+i(\mu+1/2)\pi} + E_4(z)\right| \geqq 1 - e^{-2} - \frac{C_5}{R}.$$

If $R > C_5/(1 - e^{-2})$, then the right hand side is positive. Similarly to (3.6), it follows that

$$|e^{-(1-c)z}| \leqq e^{-(1-c)\eta}, \quad |e^{-2cz+i(\mu+1/2)\pi}| \leqq e^{-2}.$$

Therefore we deduce from (3.7) that

$$\begin{aligned}\left|\frac{I_\mu(cz)}{I_\mu(z)}\right| &\leqq \frac{1}{\sqrt{c}}|e^{-(1-c)z}|\frac{|1+e^{-2cz-i(\mu+1/2)\pi}+E_4(cz)|}{|1+e^{-2z-i(\mu+1/2)\pi}+E_4(z)|} \\ &\leqq \frac{1}{\sqrt{c}}e^{-(1-c)\eta}\frac{1+e^{-2}+C_5/cR}{1-e^{-2}-C_5/R}\end{aligned} \tag{3.8}$$

for $R > C_5/(1-e^{-2})$.

We next consider the case where $-\pi/2+\delta \leqq \arg z \leqq 0$. By virtue of (3.2), (3.3), (3.4) and (3.5), we have

$$\begin{aligned}I_\mu(z) &= \frac{1}{2}e^{-i\pi\mu/2}\{H_\mu^{(1)}(ze^{i\pi/2})+H_\mu^{(2)}(ze^{i\pi/2})\} \\ &= \frac{e^z}{\sqrt{2\pi z}}\left[1+E_2(ze^{i\pi/2})+e^{-2z-i(\mu+1/2)\pi}\{1+E_1(ze^{i\pi/2})\}\right].\end{aligned}$$

If we write

$$I_\mu(z) = \frac{e^z}{\sqrt{2\pi z}}\left\{1+e^{-2z+i(\mu+1/2)\pi}+E_5(z)\right\},$$

then we have $|E_5(z)| \leqq C_5/|z|$ similarly to (3.7). Hence it follows from (3.6) that it has the same estimate as (3.8) This completes the proof of this lemma. □

Next we show an estimate for the ratio of J_μ.

Lemma 3.2. *Let c and η be real numbers with $0 < c < 1$ and $\eta \geqq 1$. Then there exist constants C_6, C_7 and $\kappa_2(c,\eta)$ such that, if $R > \kappa_2(c,\eta)$ and $J_\mu(R) \neq 0$,*

$$\left|\frac{J_\mu(cz)}{J_\mu(z)}\right|^2 \leqq \frac{2e^{2c\eta}+2+C_6/R}{c(2-C_7/R)}$$

holds for every $z = Re^{i\theta}$ with $|\theta| \leqq \mathrm{Arcsin}(\eta/R)$ and $\sin(2R\cos\theta-\pi\mu) > 0$.

Proof. For simplicity we write δ for $\mathrm{Arcsin}(\eta/R)$. Let $z = Re^{i\theta}$ and $|\theta| \leqq \delta$. Since $0 < \delta < \pi/6$ for $R > 2\eta$, we have that

$$|\sin\theta| \leqq \sin\delta = \frac{\eta}{R}. \tag{3.9}$$

It follows from $|e^{\pm i(z-\pi\mu/2-\pi/4)}| = e^{\mp R\sin\theta}$ that

$$e^{-\eta} \leqq |e^{\pm i(z-\pi\mu/2-\pi/4)}| \leqq e^{\eta} \tag{3.10}$$

for $R > 2\eta$. Therefore we deduce from (3.3), (3.4) and (3.5) that

$$J_\mu(z) = \frac{1}{\sqrt{2\pi z}}\left\{e^{i(z-\pi\mu/2-\pi/4)}+e^{-i(z-\pi\mu/2-\pi/4)}+R_6(z)\right\}, \tag{3.11}$$

where $|E_6(z)| \leqq C_8/|z|$. Hence we obtain from (3.10)

$$\begin{aligned}|J_\mu(z)|^2 = \frac{1}{2\pi R}\{&e^{2R\sin\theta}+e^{-2R\sin\theta} \\ &+2\cos(2R\cos\theta-\pi\mu-\pi/2)+E_7(z)\},\end{aligned} \tag{3.12}$$

where $|E_7(z)| \leqq C_9/|z|$. Combining with (3.9), we get

$$|J_\mu(cz)|^2 \leqq \frac{1}{2\pi cR}\left(2e^{2c\eta} + 2 + \frac{C_9}{cR}\right).$$

Finally we show a lower bound of $|J_\mu(z)|^2$. Note that

$$\cos(2R\cos\theta - \pi\mu - \pi/2) = \sin(2R\cos\theta - \pi\mu) > 0.$$

Using the elementary inequality $x + y \geqq 2\sqrt{xy}$ for $x, y \geqq 0$, we can obtain

$$e^{2R\sin\theta} + e^{-2R\sin\theta} + 2\cos(2R\cos\theta - \pi\mu - \pi/2) > 2.$$

Hence the right hand side of (3.12) is not larger than

$$\frac{2 - |E_7(z)|}{2\pi R} \geqq \frac{2 - C_9/R}{2\pi R}.$$

If $R > 2/C_8$, the right hand side is positive. □

4. The first hitting time in the case of $0 < a < b$.

From now on, for a suitable function f, the notation $\mathcal{L}[f]$ implies the Laplace transform of f and the inverse Laplace transform of f is denoted by $\mathcal{L}^{-1}[f]$. All formulae concerning Laplace and inverse Laplace transforms can be found in [13].

This section is devoted to a proof of Theorem 2.1. For $t > 0$ and $\nu \in \mathbb{R}$ let

$$F_{a,b}^{(\nu)}(t) = P(\tau_{a,b}^{(\nu)} \leqq t).$$

A standard formula shows that

$$\mathcal{L}[F_{a,b}^{(\nu)}](\lambda) = \frac{1}{\lambda}E[e^{\tau_{a,b}^{(\nu)}}](\lambda)$$

for $\lambda > 0$. For simplicity we put $G_{a,b}^{(\nu)}(t) = F_{a,b}^{(\nu)}(2b^2t)$. Then we have

$$\mathcal{L}[G_{a,b}^{(\nu)}](\lambda) = \frac{1}{2b^2}\mathcal{L}[F_{a,b}^{(\nu)}]\left(\frac{\lambda}{2b^2}\right).$$

Hence it follows from (2.2) and (2.3) that

$$\mathcal{L}[G_{a,b}^{(\nu)}](\lambda) = \begin{cases} \dfrac{1}{\alpha^\nu}\dfrac{I_\nu(\alpha\sqrt{\lambda})}{\lambda I_\nu(\sqrt{\lambda})} & \text{if } \nu > -1, \\[2ex] \dfrac{1}{\alpha^\nu}\dfrac{I_{-\nu}(\alpha\sqrt{\lambda})}{\lambda I_{-\nu}(\sqrt{\lambda})} & \text{if } \nu \leqq -1, \end{cases} \tag{4.1}$$

where $\alpha = a/b$. Note that $0 < \alpha < 1$. In order to calculate $G_{a,b}^{(\nu)}(t)$, we need to consider $I_\mu(\alpha x)/I_\mu(x)$ for $x > 0$ and $\mu > -1$.

Let $\mu > -1$. Recall the formula

$$I_\mu(z) = \sum_{n=0}^\infty \frac{(z/2)^{\mu+2n}}{n!\Gamma(n+\mu+1)} \tag{4.2}$$

for $z \in D$. For $0 < a \leqq 1$ and $z \in \mathbb{C}$ we define

$$\varphi_{\mu,a}(z) = a^\mu \sum_{n=0}^\infty \frac{(az)^{2n}}{2^{\mu+2n} n!\Gamma(n+\mu+1)}.$$

Since $\varphi_{\mu,a}(ze^{im\pi}) = \varphi_{\mu,a}(z)$ for any $m \in \mathbb{Z}$, we have that $\varphi_{\mu,a}$ is a single-valued holomorphic function on $\mathbb{C}$ and an extension of $z^{-\mu}I_\mu(az)$, which means that $\varphi_{\mu,a}(z)$ coincides with $z^{-\mu}I_\mu(az)$ for any $z \in D$. Recall that all zeros of I_μ are on the imaginary line. Hence zeros of $\varphi_{\mu,1}$ coincide with those of I_μ including multiplicities. This yields that the function $\varphi_{\mu,a}/\varphi_{\mu,1}$ on $\mathbb{C}$ is single-valued, meromorphic and an extension of $I_\mu(az)/I_\mu(z)$.

For $c \in (0,1)$ and $w \in D$ with $I_\mu(w) \neq 0$, we set

$$f^w_{\mu,c}(z) = \frac{w\varphi_{\mu,c}(z)}{z(z-w)\varphi_{\mu,1}(z)}, \qquad z \in \mathbb{C} \setminus \{0, w\},\ I_\mu(z) \neq 0$$

and

$$\Xi(R) = \frac{1}{2\pi i}\int_{C(R)} f^w_{\mu,c}(z)dz$$

for $R > 0$, where $C(R) = \{z \in \mathbb{C}\,;\, z = Re^{i\theta},\ -\pi < \theta \leqq \pi\}$. The singular points of $f^w_{\mu,c}$ are 0, w and $j_{\mu,k}e^{\pm i\pi/2}$ for $k \geqq 1$ and they are all poles of order 1. In order to apply the residue theorem to $\Xi(R)$, we need to consider the zeros of $\varphi_{\mu,1}$ inside $C(R)$ and to take R such that a zero of $\varphi_{\mu,1}$ is not on $C(R)$, but it is sufficient to put

$$R_n = \left(n + \frac{1}{2}\mu + \frac{1}{4}\right)\pi.$$

because of the following. It is known (cf. [15], p.506) that the large zeros of J_μ are given by the asymptotic expansion

$$\left(n + \frac{1}{2}\mu - \frac{1}{4}\right)\pi - \frac{4\mu^2 - 1}{8(n+\mu/2-1/4)\pi} - \frac{(4\mu^2-1)(28\mu^2-31)}{384\{(n+\mu/2-1/4)\pi\}^3} - \cdots.$$

In particular, for any $\varepsilon \in (0, \pi/4)$ there exists an integer $n_1 \geqq 1$ such that, for $n \geqq n_1$

$$\left| j_{\mu,n} - \left(n + \frac{1}{2}\mu - \frac{1}{4}\right)\pi \right| < \varepsilon, \tag{4.3}$$

which yields

$$j_{\mu,n} + \varepsilon < R_n < j_{\mu,n+1} - \varepsilon. \tag{4.4}$$

Taking the two lemmas in the previous section into account, we take n so large that (4.4) and

$$R_n > \max\left\{|w|, \kappa_1(c,1), \kappa_2(c,1), \frac{8}{\pi}\right\}$$

hold. Then we deduce from the residue theorem

$$\begin{aligned}\varXi(R_n) = &\operatorname{Res}(0; f^w_{\mu,c}) + \operatorname{Res}(w; f^w_{\mu,c}) \\ &+ \sum_{k=1}^n \operatorname{Res}(j_{\mu,k}e^{i\pi/2}; f^w_{\mu,c}) + \sum_{k=1}^n \operatorname{Res}(j_{\mu,k}e^{-i\pi/2}; f^w_{\mu,c}),\end{aligned}$$

where $\operatorname{Res}(v; f)$ is the residue of a function f at a pole v.

It is easy to see that

$$\operatorname{Res}(0; f^w_{\mu,c}) = -\frac{\varphi_{\mu,c}(0)}{\varphi_{\mu,1}(0)} = -c^\mu$$

and that

$$\operatorname{Res}(w; f^w_{\mu,c}) = \frac{\varphi_{\mu,c}(w)}{\varphi_{\mu,1}(w)} = \frac{I_\mu(cw)}{I_\mu(w)}.$$

Moreover we have

$$\operatorname{Res}(j_{\mu,k}e^{i\pi/2}; f^w_{\mu,c}) = \frac{w\varphi_{\mu,c}(j_{\mu,k}e^{i\pi/2})}{j_{\mu,k}e^{i\pi/2}(j_{\mu,k}e^{i\pi/2} - w)\varphi'_{\mu,1}(j_{\mu,k}e^{i\pi/2})} \tag{4.5}$$

for each $k \geqq 1$. Recall

$$\varphi'_{\mu,1}(z) = \frac{d}{dz}\{z^{-\mu}I_\mu(z)\} = z^{-\mu}I_{\mu+1}(z)$$

for $z \in D$. Then, since

$$\frac{\varphi_{\mu,c}(z)}{\varphi'_{\mu,1}(z)} = \frac{I_\mu(cz)}{I_{\mu+1}(z)},$$

we obtain

$$\operatorname{Res}(j_{\mu,k}e^{i\pi/2}; f^w_{\mu,c}) = \frac{w}{j_{\mu,k}e^{i\pi/2}(j_{\mu,k}e^{i\pi/2} - w)} \frac{I_\mu(cj_{\mu,k}e^{i\pi/2})}{I_{\mu+1}(j_{\mu,k}e^{i\pi/2})}.$$

From (3.1) and (3.2) we deduce

$$\frac{I_\mu(cj_{\mu,k}e^{i\pi/2})}{I_{\mu+1}(j_{\mu,k}e^{i\pi/2})} = e^{-i\pi/2}\frac{J_\mu(cj_{\mu,k})}{J_{\mu+1}(j_{\mu,k})}$$

and conclude

$$\operatorname{Res}(j_{\mu,k}e^{i\pi/2}; f^w_{\mu,c}) = -\frac{w}{j_{\mu,k}(j_{\mu,k}e^{i\pi/2} - w)} \frac{J_\mu(cj_{\mu,k})}{J_{\mu+1}(j_{\mu,k})}. \tag{4.6}$$

Similarly to (4.6), we have that

$$\operatorname{Res}(j_{\mu,k}e^{-i\pi/2}; f^w_{\mu,c}) = -\frac{w}{j_{\mu,k}(j_{\mu,k}e^{-i\pi/2} - w)} \frac{J_\mu(cj_{\mu,k})}{J_{\mu+1}(j_{\mu,k})}. \tag{4.7}$$

Therefore, summing up the right hand sides of (4.6) and (4.7), we obtain

$$\varXi(R_n) = -c^{\mu} + \frac{I_\mu(cw)}{I_\mu(w)} + \sum_{k=1}^{n} \frac{2w^2}{j_{\mu,k}(w^2+j_{\mu,k}^2)} \frac{J_\mu(cj_{\mu,k})}{J_{\mu+1}(j_{\mu,k})}. \tag{4.8}$$

We next prove that $\varXi(R_n)$ tends to 0 as $n\to\infty$. We have

$$\varXi(R_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{w}{R_n e^{i\theta}-w} \frac{\varphi_{\mu,c}(R_n e^{i\theta})}{\varphi_{\mu,1}(R_n e^{i\theta})} d\theta,$$

which is equal to the summation of the following three integrals:

$$\begin{aligned}
\varXi_1(R_n) &= \frac{w}{2\pi}\int_{-\pi}^{-\pi/2} \frac{1}{R_n e^{i\theta}-w} \frac{\varphi_{\mu,c}(R_n e^{i\theta})}{\varphi_{\mu,1}(R_n e^{i\theta})} d\theta, \\
\varXi_2(R_n) &= \frac{w}{2\pi}\int_{-\pi/2}^{\pi/2} \frac{1}{R_n e^{i\theta}-w} \frac{\varphi_{\mu,c}(R_n e^{i\theta})}{\varphi_{\mu,1}(R_n e^{i\theta})} d\theta, \\
\varXi_3(R_n) &= \frac{w}{2\pi}\int_{\pi/2}^{\pi} \frac{1}{R_n e^{i\theta}-w} \frac{\varphi_{\mu,c}(R_n e^{i\theta})}{\varphi_{\mu,1}(R_n e^{i\theta})} d\theta.
\end{aligned}$$

Recall $\varphi_{\mu,a}(ze^{im\pi}) = \varphi_{\mu,a}(z)$ for $0<a\leqq 1$ and $m\in\mathbb{Z}$. Then we have

$$\begin{aligned}
\varXi_1(R_n) &= \frac{w}{2\pi}\int_0^{\pi/2} \frac{1}{R_n e^{i(\theta-\pi)}-w} \frac{\varphi_{\mu,c}(R_n e^{i(\theta-\pi)})}{\varphi_{\mu,1}(R_n e^{i(\theta-\pi)})} d\theta \\
&= -\frac{w}{2\pi}\int_0^{\pi/2} \frac{1}{R_n e^{i\theta}+w} \frac{\varphi_{\mu,c}(R_n e^{i\theta})}{\varphi_{\mu,1}(R_n e^{i\theta})} d\theta.
\end{aligned}$$

Similarly to $\varXi_1(R_n)$, we can show

$$\begin{aligned}
\varXi_3(R_n) &= \frac{w}{2\pi}\int_{\pi/2}^{0} \frac{1}{R_n e^{i(\theta+\pi)}-w} \frac{\varphi_{\mu,c}(R_n e^{i(\theta+\pi)})}{\varphi_{\mu,1}(R_n e^{i(\theta+\pi)})} d\theta \\
&= -\frac{w}{2\pi}\int_{-\pi/2}^{0} \frac{1}{R_n e^{i\theta}+w} \frac{\varphi_{\mu,c}(R_n e^{i\theta})}{\varphi_{\mu,1}(R_n e^{i\theta})} d\theta.
\end{aligned}$$

Hence we get

$$\begin{aligned}
\varXi_1(R_n)+\varXi_3(R_n) &= -\frac{w}{2\pi}\int_{-\pi/2}^{\pi/2} \frac{1}{R_n e^{i\theta}+w} \frac{\varphi_{\mu,c}(R_n e^{i\theta})}{\varphi_{\mu,1}(R_n e^{i\theta})} d\theta \\
&= -\frac{w}{2\pi}\int_{-\pi/2}^{\pi/2} \frac{1}{R_n e^{i\theta}+w} \frac{I_\mu(cR_n e^{i\theta})}{I_\mu(R_n e^{i\theta})} d\theta
\end{aligned}$$

and

$$\varXi(R_n) = \frac{w^2}{\pi}\int_{-\pi/2}^{\pi/2} \frac{1}{R_n^2 e^{i2\theta}-w^2} \frac{I_\mu(cR_n e^{i\theta})}{I_\mu(R_n e^{i\theta})} d\theta.$$

Let

$$\delta_n = \operatorname{Arcsin}\frac{1}{R_n}.$$

Then $0 < \delta_n < \pi/2$. It follows that $\Xi(R_n)$ is equal to the sum of the following three integrals:

$$\widetilde{\Xi}_1(R_n) = \frac{w^2}{\pi}\int_{-\pi/2+\delta_n}^{\pi/2-\delta_n} \frac{1}{R_n^2 e^{i2\theta} - w^2}\frac{I_\mu(cR_n e^{i\theta})}{I_\mu(R_n e^{i\theta})}d\theta,$$
$$\widetilde{\Xi}_2(R_n) = \frac{w^2}{\pi}\int_{\pi/2-\delta_n}^{\pi/2} \frac{1}{R_n^2 e^{i2\theta} - w^2}\frac{I_\mu(cR_n e^{i\theta})}{I_\mu(R_n e^{i\theta})}d\theta,$$
$$\widetilde{\Xi}_3(R_n) = \frac{w^2}{\pi}\int_{-\pi/2}^{-\pi/2+\delta_n} \frac{1}{R_n^2 e^{i2\theta} - w^2}\frac{I_\mu(cR_n e^{i\theta})}{I_\mu(R_n e^{i\theta})}d\theta.$$

By (4.4), we have $J_\mu(R_n) \neq 0$. Applying Lemma 3.1 for $\eta = 1$ and $R = R_n$, we obtain

$$|\widetilde{\Xi}_1(R_n)| \leqq \frac{|w|^2}{\pi}\frac{1}{R_n^2 - |w|^2}\frac{1}{\sqrt{c}}e^{-2(1-c)}\frac{2 + C_3/R_n}{1 - e^{-2} - C_4/R_n}(\pi - 2\delta_n).$$

This yields that $\widetilde{\Xi}_1(R_n)$ tends to 0 as $n \to \infty$.

About $\widetilde{\Xi}_2(R_n)$, it holds that

$$\begin{aligned}|\Xi_2(R_n)| &\leqq \frac{|w|^2}{\pi}\frac{1}{R_n^2 - |w|^2}\int_{\pi/2-\delta_n}^{\pi/2}\left|\frac{I_\mu(cR_n e^{i\theta})}{I_\mu(R_n e^{i\theta})}\right|d\theta \\ &= \frac{|w|^2}{\pi}\frac{1}{R_n^2 - |w|^2}\int_{-\delta_n}^{0}\left|\frac{I_\mu(cR_n e^{i\theta}e^{i\pi/2})}{I_\mu(R_n e^{i\theta}e^{i\pi/2})}\right|d\theta.\end{aligned}$$

By (3.1) and (3.2), we have

$$\int_{-\delta_n}^{0}\left|\frac{I_\mu(cR_n e^{i\theta}e^{i\pi/2})}{I_\mu(R_n e^{i\theta}e^{i\pi/2})}\right|d\theta = \int_{-\delta_n}^{0}\left|\frac{J_\mu(cR_n e^{i\theta})}{J_\mu(R_n e^{i\theta})}\right|d\theta.$$

In order to apply Lemma 3.2 for $\eta = 1$ and $R = R_n$, we need to see that

$$\sin(2R_n\cos\theta - \pi\mu) > 0 \tag{4.9}$$

for $|\theta| \leqq \delta_n$. It is obvious that

$$2R_n\cos\theta - \pi\mu \leqq 2R_n - \pi\mu = 2n\pi + \frac{1}{2}\pi.$$

Since $|\theta| \leqq \delta_n$ and $\sin\delta_n = 1/R_n$, it is easy to show

$$2R_n\left(1 - \frac{1}{R_n^2}\right) - \pi\mu \leqq 2R_n\sqrt{1 - \frac{1}{R_n^2}} - \pi\mu \leqq 2R_n\cos\theta - \pi\mu.$$

Since $R_n > 8/\pi$, we have

$$2n\pi + \frac{1}{4}\pi \leqq 2R_n\cos\theta - \pi\mu \leqq 2n\pi + \frac{1}{2}\pi$$

and (4.9). Now, applying Lemma 3.2, we obtain

$$\widetilde{\Xi}_2(R_n) \leqq \frac{C_{10}\delta_n}{R_n^2 - |w|^2}$$

for some constant C_{10}. This immediately implies that $\widetilde{\Xi}_2(R_n)$ tends to 0 as $n \to \infty$.

For $\widetilde{\Xi}_3(R_n)$, since

$$|\Xi_3(R_n)| \leqq \frac{|w|^2}{\pi} \frac{1}{R_n^2 - |w|^2} \int_0^{\delta_n} \left| \frac{I_\mu(cR_n e^{i\theta} e^{-i\pi/2})}{I_\mu(R_n e^{i\theta} e^{-i\pi/2})} \right| d\theta,$$

we can conclude that $\Xi_3(R_n)$ tends to 0 as $n \to \infty$ in a similar way as to $\Xi_2(R_n)$. Accordingly $\Xi(R_n)$ tends to 0 as $n \to \infty$ and the right hand side of (4.8) converges. Therefore we have obtained the following.

Theorem 4.1. *If $\mu > -1$ and $0 < c < 1$, then*

$$\frac{I_\mu(cw)}{I_\mu(w)} = c^\mu - \sum_{k=1}^{\infty} \frac{2w^2}{j_{\mu,k}(w^2 + j_{\mu,k}^2)} \frac{J_\mu(cj_{\mu,k})}{J_{\mu+1}(j_{\mu,k})} \tag{4.10}$$

for $w \in D$ with $I_\mu(w) \neq 0$.

Recall that $\alpha = a/b \in (0,1)$ and that all zeros of I_μ are pure imaginary for $\mu > -1$. Theorem 4.1 implies that $\mathcal{L}[G_{a,b}^{(\nu)}]$ can be represented by the infinite sum of rational functions. Indeed, it follows from (4.1) and (4.10) that, for $\lambda > 0$ if $\nu > -1$,

$$\mathcal{L}[G_{a,b}^{(\nu)}](\lambda) = \frac{1}{\lambda} - \frac{2}{\alpha^\nu} \sum_{k=1}^{\infty} \frac{1}{j_{\nu,k}(\lambda + j_{\nu,k}^2)} \frac{J_\nu(\alpha j_{\nu,k})}{J_{\nu+1}(j_{\nu,k})} \tag{4.11}$$

and that, for $\lambda > 0$ if $\nu \leqq -1$,

$$\mathcal{L}[G_{a,b}^{(\nu)}](\lambda) = \frac{1}{\alpha^{2\nu}\lambda} - \frac{2}{\alpha^\nu} \sum_{k=1}^{\infty} \frac{1}{j_{-\nu,k}(\lambda + j_{-\nu,k}^2)} \frac{J_{-\nu}(\alpha j_{-\nu,k})}{J_{-\nu+1}(j_{-\nu,k})}. \tag{4.12}$$

In order to prove Theorem 2.1, we need to see that, if $\mu > -1$, there exists a positive constant C_{11} such that

$$\left| \frac{J_\mu(\alpha j_{\mu,k})}{J_{\mu+1}(j_{\mu,k})} \right| \leqq C_{11} \tag{4.13}$$

for any $k \geqq 1$. By formula (3.11) we get

$$J_\mu(x) = \sqrt{\frac{2}{\pi x}} \left\{ \cos\left(x - \frac{1}{2}\pi\mu - \frac{1}{4}\pi \right) + E_8^{(\mu)}(x) \right\}$$

for $x > 0$. Here $|E_8^{(\mu)}(x)| \leqq C_{12}^{(\mu)}/x$ for some constant $C_{12}^{(\mu)}$, which is independent of x. Therefore we obtain

$$\frac{J_\mu(\alpha j_{\mu,n})}{J_{\mu+1}(j_{\mu,n})} = \frac{1}{\sqrt{\alpha}} \frac{\cos(\alpha j_{\mu,n} - \pi\mu/2 - \pi/4) + E_8^{(\mu)}(\alpha j_{\mu,n})}{\cos\{(j_{\mu,n} - \pi(\mu+1)/2 - \pi/4\} + E_8^{(\mu+1)}(j_{\mu,n})}. \tag{4.14}$$

Let $\varepsilon \in (0, \pi/4)$. It follows from (4.3) that

$$\left|\left(j_{\mu,n} - \frac{1}{2}\pi\mu - \frac{3}{4}\pi\right) - (n-1)\pi\right| < \varepsilon$$

for $n \geqq n_1$ and that

$$\left|\cos\left\{j_{\mu,n} - \frac{1}{2}\pi(\mu+1) - \frac{1}{4}\pi\right\}\right| \geqq \cos\varepsilon \geqq \frac{1}{\sqrt{2}} \tag{4.15}$$

for $n \geqq n_1$. We take n so large that

$$\frac{C_{12}^{(\mu)}}{\alpha j_{\mu,n}} \leqq 1, \quad \frac{C_{12}^{(\mu+1)}}{j_{\mu,n}} \leqq \frac{1}{2\sqrt{2}}$$

and (4.15) hold. Then we obtain from (4.14)

$$\left|\frac{J_\mu(\alpha j_{\mu,n})}{J_{\mu+1}(j_{\mu,n})}\right| \leqq \frac{4\sqrt{2}}{\sqrt{\alpha}},$$

which implies (4.13).

We are ready to complete our proof of Theorem 2.1. It follows from (4.3) and (4.13) that

$$\sum_{k=1}^{\infty}\left|\frac{J_\mu(\alpha j_{\mu,k})}{j_{\mu,k}J_{\mu+1}(j_{\mu,k})}\right| \int_0^\infty e^{-\lambda t} e^{-j_{\mu,k}^2 t} dt \leqq C_{11}\sum_{k=1}^{\infty}\frac{1}{j_{\mu,k}(\lambda + j_{\mu,k}^2)} < \infty$$

for each $\lambda > 0$. This yields

$$\begin{aligned}\int_0^\infty e^{-\lambda t}\sum_{k=1}^{\infty}\frac{J_\mu(\alpha j_{\mu,k})}{j_{\mu,k}J_{\mu+1}(j_{\mu,k})}e^{-j_{\mu,k}^2 t}dt &= \sum_{k=1}^{\infty}\frac{J_\mu(\alpha j_{\mu,k})}{j_{\mu,k}J_{\mu+1}(j_{\mu,k})}\int_0^\infty e^{-\lambda t - j_{\mu,k}^2 t}dt \\ &= \sum_{k=1}^{\infty}\frac{J_\mu(\alpha j_{\mu,k})}{j_{\mu,k}J_{\mu+1}(j_{\mu,k})}\frac{1}{\lambda + j_{\mu,k}^2}.\end{aligned}$$

Therefore we deduce from (4.11) and (4.12) that, if $\nu > -1$,

$$G_{a,b}^{(\nu)}(t) = 1 - \frac{2}{\alpha^\nu}\sum_{k=1}^{\infty}\frac{J_\nu(\alpha j_{\nu,k})}{j_{\nu,k}J_{\nu+1}(j_{\nu,k})}e^{-j_{\nu,k}^2 t}$$

and that, if $\nu \leqq -1$,

$$G_{a,b}^{(\nu)}(t) = \frac{1}{\alpha^{2\nu}} - \frac{2}{\alpha^\nu}\sum_{k=1}^{\infty}\frac{J_{-\nu}(\alpha j_{-\nu,k})}{j_{-\nu,k}J_{-\nu+1}(j_{-\nu,k})}e^{-j_{-\nu,k}^2 t}.$$

Recalling $G_{a,b}^{(\nu)}(t) = F_{a,b}^{(\nu)}(2b^2 t)$ for $t > 0$, we complete our proof of Theorem 2.1.

Since (4.13) assures the termwise differentiation with respect to t in (2.8) and (2.9), we obtain the following expression for the densities of $\tau_{a,b}^{(\nu)}$

Corollary 4.2. *Let $0 < a < b$. We have that, if $\nu > -1$,*

$$\rho_{a,b}^{(\nu)}(t) = \frac{b^{\nu-2}}{a^\nu}\sum_{k=1}^{\infty}\frac{j_{\nu,k}J_\nu(aj_{\nu,k}/b)}{J_{\nu+1}(j_{\nu,k})}e^{-\frac{j_{\nu,k}^2 t}{2b^2}}$$

and that, if $\nu \leqq -1$,

$$\rho_{a,b}^{(\nu)}(t) = \frac{b^{\nu-2}}{a^\nu}\sum_{k=1}^{\infty}\frac{j_{-\nu,k}J_{-\nu}(aj_{-\nu,k}/b)}{J_{-\nu+1}(j_{-\nu,k})}e^{-\frac{j_{-\nu,k}^2 t}{2b^2}}.$$

5. The first hitting time in the case of $0 < b < a$.

This section devoted to a proof of Theorem 2.2. We again use the same notation $F_{a,b}^{(\nu)}$ and $G_{a,b}^{(\nu)}(t)$ as those in the previous section. Set $\alpha = a/b > 1$. It follows from (2.5) that, for $\lambda > 0$

$$\mathcal{L}[G_{a,b}^{(\nu)}](\lambda) = \frac{1}{\alpha^\nu} \frac{K_\nu(\alpha\sqrt{\lambda})}{\lambda K_\nu(\sqrt{\lambda})}.$$

Since $K_\nu = K_{-\nu}$ for $\nu \geqq 0$, it is sufficient to consider the case where $\nu \geqq 0$.

Let $\nu \geqq 0$ and $c > 1$. We first assume that $\nu - 1/2$ is an integer. In this case, there is a suitable polynomial ψ_ν of order $\nu - 1/2$ on $\mathbb{C}$ such that $\psi_\nu(0) \neq 0$ and

$$z^\nu K_\nu(z) = \sqrt{\frac{\pi}{2}} e^{-z} \psi_\nu(z) \tag{5.1}$$

(cf. [12, 15]). For example,

$$\psi_{1/2}(z) = 1, \quad \psi_{3/2}(z) = 1 + z, \quad \psi_{5/2}(z) = 3 + 3z + z^2.$$

The function $z^\nu K_\nu(z)$ is extended to a entire function and all zeros of ψ_ν are the same as those of K_ν. For $z \in \mathbb{C}$ let

$$\psi_{\nu,c}(z) = \frac{e^{-(c-1)z}\psi_\nu(cz)}{c^\nu \psi_\nu(z)}.$$

Then $\psi_{\nu,c}$ is a single-valued meromorphic function on $\mathbb{C}$ and it holds that

$$\psi_{\nu,c}(z) = \frac{K_\nu(cz)}{K_\nu(z)}$$

for $z \in D$. Therefore, if $z \in \mathbb{C}$ is not a zero of K_ν, we have

$$\psi_{\nu,c}(z) = \lim_{v \to z} \frac{K_\nu(cv)}{K_\nu(v)},$$

which implies that $K_\nu(cx)/K_\nu(x)$ can be determined uniquely for $x < 0$ if x is not a zero of K_ν.

Recall our notation $z_{\nu,1}, \ldots, z_{\nu,N(\nu)}$ for the zeros of K_ν. Let w is a point in D with $K_\nu(w) \neq 0$. We take R so large that w and all zeros of K_ν are inside $C(R)$, a circle whose center is the origin and radius R.

We set

$$\Theta(R) = \frac{1}{2\pi i} \int_{C(R)} g_{\nu,c}^w(z) dz, \tag{5.2}$$

where

$$g_{\nu,c}^w(z) = \frac{we^{(c-1)z}\psi_{\nu,c}(z)}{z(z-w)}$$

for $z \in \mathbb{C}$. The singular points of $g_{\nu,c}^w$ are 0, w and zeros of K_ν, which are all poles of order 1. The residue theorem yields that, if $N(\nu) = 0$,

$$\Theta(R) = \mathrm{Res}(0; g_{\nu,c}^w) + \mathrm{Res}(w; g_{\nu,c}^w)$$

and that, if $N(\nu) \geqq 1$,

$$\Theta(R) = \mathrm{Res}(0; g^w_{\nu,c}) + \mathrm{Res}(w; g^w_{\nu,c}) + \sum_{j=1}^{N(\nu)} \mathrm{Res}(z_{\nu,j}; g^w_{\nu,c}).$$

By definition of the function $g^w_{\nu,c}$, we have

$$\mathrm{Res}(0; g^w_{\nu,c}) = -\psi_{\nu,c}(0) = -\frac{1}{c^\nu}$$

and

$$\mathrm{Res}(w; g^w_{\nu,c}) = e^{(c-1)w}\psi_{\nu,c}(w) = e^{(c-1)w}\frac{K_\nu(cw)}{K_\nu(w)}.$$

If $N(\nu) \geqq 1$, the residue of $g^w_{\nu,c}$ at $z_{\nu,j}$ is equal to

$$\lim_{z\to z_{\nu,j}} \frac{w\psi_\nu(cz)}{c^\nu z(z-w)}\frac{z-z_{\nu.j}}{\psi_\nu(z)} = \frac{w\psi_\nu(cz_{\nu,j})}{c^\nu z_{\nu,j}(z_{\nu,j}-w)\psi'_\nu(z_{\nu,j})}$$

for $1 \leqq j \leqq N(\nu)$. Since Lemma 3.1 in [9] gives that, if $\psi_\nu(z_0) = 0$,

$$\psi_{\nu+1}(z_0) = -z_0\psi'_\nu(z_0),$$

we obtain

$$\mathrm{Res}(z_{\nu,j}; g^w_{\nu,c}) = -\frac{w\psi_\nu(cz_{\nu,j})}{c^\nu(z_{\nu,j}-w)\psi_{\nu+1}(z_{\nu,j})}. \tag{5.3}$$

If $z \in D$, it follows from (5.1) that

$$\frac{K_\nu(cz)}{K_{\nu+1}(z)} = \frac{ze^{-(c-1)z}\psi_\nu(cz)}{c^\nu\psi_{\nu+1}(z)}. \tag{5.4}$$

Then $K_\nu(cz)/K_{\nu+1}(z)$ is extended to a meromorphic function on $\mathbb{C}$. This implies that $K_\nu(cx)/K_{\nu+1}(x)$ can be determined uniquely for $x < 0$ with $K_{\nu+1}(x) \neq 0$. From (5.3) and (5.4) we deduce

$$\mathrm{Res}(z_{\nu,j}; g^w_{\nu,c}) = -\frac{we^{(c-1)z_{\nu,j}}}{z_{\nu,j}(z_{\nu,j}-w)}\frac{K_\nu(cz_{\nu,j})}{K_{\nu+1}(z_{\nu,j})}$$

and

$$\Theta(R) = -\frac{1}{c^\nu} + e^{(c-1)w}\frac{K_\nu(cw)}{K_\nu(w)} - \sum_{j=1}^{N(\nu)} \frac{we^{(c-1)z_{\nu,j}}}{z_{\nu,j}(z_{\nu,j}-w)}\frac{K_\nu(cz_{\nu,j})}{K_{\nu+1}(z_{\nu,j})}.$$

$\Theta(R)$ tends to 0 as $R \to \infty$ since $g^w_{\nu,c}(z) = O(|z|^{-2})$. Hence we obtain

$$\frac{K_\nu(cw)}{K_\nu(w)} = \begin{cases} \dfrac{e^{-(c-1)w}}{c^\nu} & \text{if } \nu = \dfrac{1}{2}, \\[2ex] \dfrac{e^{-(c-1)w}}{c^\nu} + \displaystyle\sum_{j=1}^{N(\nu)} \frac{we^{(c-1)(z_{\nu,j}-w)}}{z_{\nu,j}(z_{\nu,j}-w)}\frac{K_\nu(cz_{\nu,j})}{K_{\nu+1}(z_{\nu,j})} & \text{if } \nu \neq \dfrac{1}{2} \end{cases} \tag{5.5}$$

in the case where $\nu - 1/2$ is a non-negative integer.

We next consider the case where $\nu - 1/2$ is not an integer and look for a nice expression for $K_\nu(cw)/K_\nu(w)$ like (5.5). If ν is not an integer, it is well-known (cf. [15], p.80) that

$$K_\nu(ze^{im\pi}) = e^{-im\pi\nu}K_\nu(z) - i\pi\frac{\sin(m\pi\nu)}{\sin(\pi\nu)}I_\nu(z)$$

for $z \in D$ and $m \in \mathbb{Z}$. When ν is an integer, we also have

$$K_\nu(ze^{im\pi}) = e^{-im\pi\nu}K_\nu(z) - i\pi m(-1)^{(m-1)\nu}I_\nu(z)$$

for $z \in D$ and $m \in \mathbb{Z}$, which is easily seen from

$$\lim_{\mu\to n} K_\mu(z) = K_n(z)$$

for each integer n. Especially, for $z \in D$, we have

$$K_\nu(ze^{i\pi}) = e^{-i\pi\nu}K_\nu(z) - i\pi I_\nu(z), \tag{5.6}$$

$$K_\nu(ze^{-i\pi}) = e^{i\pi\nu}K_\nu(z) + i\pi I_\nu(z). \tag{5.7}$$

It follows from these identities that the function $K_\nu(cz)/K_\nu(z)$ can not be extended to a meromorphic function on $\mathbb{C}$. For $z \in D$ let

$$h^w_{\nu,c}(z) = \frac{we^{(c-1)z}}{z(z-w)}\frac{K_\nu(cz)}{K_\nu(z)}.$$

In order to give a formula for $K_\nu(cw)/K_\nu(w)$ like (5.5), we consider the integral of $h^w_{\nu,c}$ on a suitable contour. However we can not adopt a circle as the contour like (5.2) since $h^w_{\nu,c}$ can not extend to a meromorphic function on $\mathbb{C}$.

Let ε and R be positive numbers with $2\varepsilon < R$. We set

$$\theta_R = \mathrm{Arcsin}\,\frac{\varepsilon}{R}.$$

As a contour, we take the curve γ defined by

$$\begin{aligned}
&\gamma_0 \,:\, z = Re^{i\theta},\ -\pi+\theta_R \leqq \theta \leqq \pi-\theta_R,\\
&\gamma_1 \,:\, z = x+i\varepsilon,\ -R\cos\theta_R \leqq x \leqq 0\\
&\gamma_2 \,:\, z = \varepsilon e^{i\theta},\ -\pi/2 \leqq \theta \leqq \pi/2,\\
&\gamma_3 \,:\, z = x-i\varepsilon,\ -R\cos\theta_R \leqq x \leqq 0\\
&\gamma = \gamma_0+\gamma_1-\gamma_2-\gamma_3.
\end{aligned}$$

We take R so large and ε so small that w and all zeros of K_ν are inside γ. Then, setting

$$\varPi(R,\varepsilon) = \frac{1}{2\pi i}\int_\gamma h^w_{\nu,c}(z)dz, \quad \varPi_\ell = \frac{1}{2\pi i}\int_{\gamma_\ell} h^w_{\nu,c}(z)dz$$

for $0 \leqq \ell \leqq 3$, we have

$$\varPi(R,\varepsilon) = \varPi_0 + \varPi_1 - \varPi_2 - \varPi_3.$$

The residue theorem yields that, if $N(\nu) = 0$,

$$\varPi(R,\varepsilon) = \mathrm{Res}(w; h^w_{\nu,c}) \tag{5.8}$$

and that, if $N(\nu) \geqq 1$,

$$\varPi(R,\varepsilon) = \mathrm{Res}(w; h^w_{\nu,c}) + \sum_{j=1}^{N(\nu)} \mathrm{Res}(z_{\nu,j}; h^w_{\nu,c}). \tag{5.9}$$

It is obvious that

$$\mathrm{Res}(w; h^w_{\nu,c}) = e^{(c-1)w} \frac{K_\nu(cw)}{K_\nu(w)}.$$

When $N(\nu) \geqq 1$, by using the formula

$$zK'_\nu(z) - \nu K_\nu(z) = -zK_{\nu+1}(z)$$

(cf. [15], p.29), we can show

$$\mathrm{Res}(z_{\nu,j}; h^w_{\nu,c}) = \frac{we^{(c-1)z_{\nu,j}}}{z_{\nu,j}(z_{\nu,j}-w)} \frac{K_\nu(cz_{\nu,j})}{K'_\nu(z_{\nu,j})} = -\frac{we^{(c-1)z_{\nu,j}}}{z_{\nu,j}(z_{\nu,j}-w)} \frac{K_\nu(cz_{\nu,j})}{K_{\nu+1}(z_{\nu,j})}.$$

It follows from (5.8) and (5.9) that, if $N(\nu) = 0$,

$$\varPi(R,\varepsilon) = e^{(c-1)w} \frac{K_\nu(cw)}{K_\nu(w)}$$

and that, if $N(\nu) \geqq 1$,

$$\varPi(R,\varepsilon) = e^{(c-1)w} \frac{K_\nu(cw)}{K_\nu(w)} - \sum_{j=1}^{N(\nu)} \frac{we^{(c-1)z_{\nu,j}}}{z_{\nu,j}(z_{\nu,j}-w)} \frac{K_\nu(cz_{\nu,j})}{K_{\nu+1}(z_{\nu,j})}.$$

In order to consider asymptotic behavior of $\varPi(R,\varepsilon)$ as $R \to \infty$, we recall the asymptotic behavior of $K_\nu(z)$ as $|z| \to \infty$ from [15], p.202: if $|\arg z| < 3\pi/2$,

$$K_\nu(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \{1 + O(|z|^{-1})\}.$$

By combining the formula

$$K_\nu(z) = \frac{1}{2} i\pi e^{i\pi\nu/2} H^{(1)}_\nu(ze^{i\pi/2})$$

with (3.3) and (3.4), we can make sure that, for $|\arg z| \leqq 3\pi/2 - \delta$

$$K_\nu(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \{1 + E_1(ze^{i\pi/2})\}. \tag{5.10}$$

Here $\delta > 0$ is arbitrary given. By virtue of (5.10), we get

$$|h^w_{\nu,c}(z)| \leqq \frac{|w|}{\sqrt{c}|z| \cdot |z-w|} \frac{1 + C_1/c|z|}{1 - C_1/|z|}$$

if $z = Re^{i\theta} \in D$ and

$$|\varPi_0| \leqq \frac{1}{2\pi} \int_{-\pi+\theta_R}^{\pi-\theta_R} |h^w_{\nu,c}(Re^{i\theta})| R d\theta \leqq \frac{|w|}{\sqrt{c}(R - |w|)} \frac{1 + C_1/cR}{1 - C_1/R},$$

which tends to 0 as $R \to \infty$.

For the integral $\varPi_1$, we have

$$\begin{aligned} \varPi_1 &= \frac{w}{2\pi i} \int_{-R\cos\theta_R}^{0} \frac{e^{(c-1)(x+i\varepsilon)}}{(x+i\varepsilon)(x+i\varepsilon-w)} \frac{K_\nu(c(x+i\varepsilon))}{K_\nu(x+i\varepsilon)} dx \\ &= \frac{w}{2\pi i} \int_0^{R\cos\theta_R} \frac{e^{(c-1)(-x+i\varepsilon)}}{(x-i\varepsilon)(x-i\varepsilon+w)} \frac{K_\nu(c(-x+i\varepsilon))}{K_\nu(-x+i\varepsilon)} dx. \end{aligned}$$

Then, using (5.6) and writing the right hand side by ξ_ν,

$$\xi_\nu(z) = e^{-i\pi\nu} K_\nu(z) - i\pi I_\nu(z),$$

we get

$$\varPi_1 = \frac{w}{2\pi i} \int_0^{R\cos\theta_R} \frac{e^{-(c-1)(x-i\varepsilon)}}{(x-i\varepsilon)(x-i\varepsilon+w)} \frac{\xi_\nu(c(x-i\varepsilon))}{\xi_\nu(x-i\varepsilon)} dx.$$

Hence, letting γ_1^0 be the line in D defined by

$$\gamma_1^0 \,:\, z = x - i\varepsilon, \ 0 \leqq x \leqq R\cos\theta_R,$$

it holds that

$$\varPi_1 = \frac{w}{2\pi i} \int_{\gamma_1^0} \frac{e^{-(c-1)z}}{z(z+w)} \frac{\xi_\nu(cz)}{\xi_\nu(z)} dz.$$

Here we define three paths as follows:

$$\begin{aligned} &\gamma_1^1 \,:\, z = \varepsilon e^{i\theta}, \ -\pi/2 \leqq \theta \leqq 0, \\ &\gamma_1^2 \,:\, z = x, \ \varepsilon \leqq x \leqq R, \\ &\gamma_1^3 \,:\, z = Re^{i\theta}, \ -\theta_R \leqq \theta \leqq 0. \end{aligned}$$

Since w is inside γ, we have that $|\operatorname{Im}(w)| > \varepsilon$ if $\operatorname{Re}(w) < 0$. Recall that there is no zero of K_ν on the real axis. Then we may apply the Cauchy integral theorem for the integral on the contour consisting of γ_1^0, γ_1^1, $\gamma_1^2$and γ_1^3 to obtain

$$\varPi_1 = \varPi_1^1 + \varPi_1^2 - \varPi_1^3,$$

where

$$\begin{aligned} \varPi_1^1 &= \frac{w}{2\pi} \int_{-\pi/2}^{0} \frac{e^{-(c-1)\varepsilon e^{i\theta}}}{\varepsilon e^{i\theta} + w} \frac{\xi_\nu(c\varepsilon e^{i\theta})}{\xi_\nu(\varepsilon e^{i\theta})} d\theta, \\ \varPi_1^2 &= \frac{w}{2\pi i} \int_{\varepsilon}^{R} \frac{e^{-(c-1)x}}{x(x+w)} \frac{\xi_\nu(cx)}{\xi_\nu(x)} dx, \\ \varPi_1^3 &= \frac{w}{2\pi} \int_{-\theta_R}^{0} \frac{e^{-(c-1)Re^{i\theta}}}{Re^{i\theta} + w} \frac{\xi_\nu(cRe^{i\theta})}{\xi_\nu(Re^{i\theta})} d\theta. \end{aligned}$$

$\varPi_1^3$ tends to 0 as $R \to \infty$. In fact, noting that $\xi_\nu(xe^{i\theta}) = K_\nu(xe^{i(\theta+\pi)})$ holds for $x > 0$, we obtain from (5.10)

$$\frac{\xi_\nu(cRe^{i\theta})}{\xi_\nu(Re^{\theta})} = \frac{1}{\sqrt{c}} e^{-(c-1)Re^{i(\theta+\pi)}} \frac{1 + E_1(cRe^{i(\theta+3\pi/2)})}{1 + E_1(Re^{i(\theta+3\pi/2)})}$$

for $|\theta| < \pi/6$, which yields

$$\left| e^{-(c-1)Re^{i\theta}} \frac{\xi_\nu(cRe^{i\theta})}{\xi_\nu(Re^{\theta})} \right| \leqq \frac{1}{\sqrt{c}} \frac{1 + C_1/cR}{1 - C_1/R} \leqq C_{13} \tag{5.11}$$

for large R and a positive constant C_{13} independent of R and θ. Since $0 < \theta_R < \pi/6$, we see $\varPi_1^3 \to 0$ as $R \to \infty$.

Furthermore (5.11) shows that the function $e^{-(c-1)x}\xi_\nu(cx)/\xi_\nu(x)$ is bounded on $[\varepsilon, \infty)$ and that $\varPi_1^2$ converges as $R \to \infty$. Therefore it holds that

$$\lim_{R\to\infty} \varPi_1 = \frac{w}{2\pi} \int_{-\pi/2}^{0} \frac{e^{-(c-1)\varepsilon e^{i\theta}}}{\varepsilon e^{i\theta} + w} \frac{\xi_\nu(c\varepsilon e^{i\theta})}{\xi_\nu(\varepsilon e^{i\theta})} d\theta + \frac{w}{2\pi i} \int_{\varepsilon}^{\infty} \frac{e^{-(c-1)x}}{x(x+w)} \frac{\xi_\nu(cx)}{\xi_\nu(x)} dx.$$

In the same way, we can show that

$$\lim_{R\to\infty} (-\varPi_3) = \frac{w}{2\pi} \int_{0}^{\pi/2} \frac{e^{-(c-1)\varepsilon e^{i\theta}}}{\varepsilon e^{i\theta} + w} \frac{\zeta_\nu(c\varepsilon e^{i\theta})}{\zeta_\nu(\varepsilon e^{i\theta})} d\theta - \frac{w}{2\pi i} \int_{\varepsilon}^{\infty} \frac{e^{-(c-1)x}}{x(x+w)} \frac{\zeta_\nu(cx)}{\zeta_\nu(x)} dx,$$

where

$$\zeta_\nu(z) = K_\nu(ze^{-i\pi}) = e^{i\pi\nu} K_\nu(z) + i\pi I_\nu(z)$$

for $z \in D$ (cf. (5.7)). Note that

$$\frac{1}{2\pi i} \left\{ \frac{\zeta_\nu(cx)}{\zeta_\nu(x)} - \frac{\xi_\nu(cx)}{\xi_\nu(x)} \right\} = \frac{\cos(\pi\nu)\{I_\nu(cx)K_\nu(x) - I_\nu(x)K_\nu(cx)\}}{\{K_\nu(x)\}^2 + \pi^2\{I_\nu(x)\}^2 + 2\pi\sin(\pi\nu)K_\nu(x)I_\nu(x)}$$

and recall that the right hand side is $L_{\nu,c}(x)$. Then we get

$$\begin{aligned}\lim_{R\to\infty} \varPi(R,\varepsilon) =& \frac{w}{2\pi} \int_{-\pi/2}^{0} \frac{e^{-(c-1)\varepsilon e^{i\theta}}}{\varepsilon e^{i\theta} + w} \frac{\xi_\nu(c\varepsilon e^{i\theta})}{\xi_\nu(\varepsilon e^{i\theta})} d\theta \\ &+ \frac{w}{2\pi} \int_{0}^{\pi/2} \frac{e^{-(c-1)\varepsilon e^{i\theta}}}{\varepsilon e^{i\theta} + w} \frac{\zeta_\nu(c\varepsilon e^{i\theta})}{\zeta_\nu(\varepsilon e^{i\theta})} d\theta \\ &- \frac{w}{2\pi} \int_{-\pi/2}^{\pi/2} \frac{e^{(c-1)\varepsilon e^{i\theta}}}{\varepsilon e^{i\theta} - w} \frac{K_\nu(c\varepsilon e^{i\theta})}{K_\nu(\varepsilon e^{i\theta})} d\theta \\ &- \int_{\varepsilon}^{\infty} \frac{we^{-(c-1)x} L_{\nu,c}(x)}{x(x+w)} dx.\end{aligned} \tag{5.12}$$

We will calculate the limit of each term of (5.12) as $\varepsilon \downarrow 0$.

Lemma 5.1. *Let $c>0$, $\nu \geqq 0$ and $|\theta|<\pi$. We have that*

$$\lim_{\varepsilon\downarrow 0}\frac{K_\nu(c\varepsilon e^{i\theta})}{K_\nu(\varepsilon e^{i\theta})}=\lim_{\varepsilon\downarrow 0}\frac{\xi_\nu(c\varepsilon e^{i\theta})}{\xi_\nu(\varepsilon e^{i\theta})}=\lim_{\varepsilon\downarrow 0}\frac{\zeta_\nu(c\varepsilon e^{i\theta})}{\zeta_\nu(\varepsilon e^{i\theta})}=\frac{1}{c^\nu}.$$

Proof. It is known that

$$K_\nu(z)=\begin{cases}\log\left(\dfrac{2}{z}\right)\{1+o(1)\} & \text{if } \nu=0,\\[2mm] \dfrac{\Gamma(\nu)}{2}\left(\dfrac{2}{z}\right)^{\nu}\{1+o(1)\} & \text{if } \nu>0\end{cases}\tag{5.13}$$

as $|z|\to 0$ in D. See [12], p.111 and [15], p.512. Then, it follows from (5.13) that

$$\frac{K_\nu(c\varepsilon e^{i\theta})}{K_\nu(\varepsilon e^{i\theta})}=\begin{cases}\dfrac{\log(2/c\varepsilon)-i\theta}{\log(2/\varepsilon)-i\theta}\dfrac{1+o(1)}{1+o(1)} & \text{if } \nu=0,\\[2mm] \dfrac{1}{c^\nu}\dfrac{1+o(1)}{1+o(1)} & \text{if } \nu>0,\end{cases}$$

which converges to $1/c^\nu$ as $\varepsilon\downarrow 0$. From (4.2) and (5.13) we deduce that $I_\nu(xe^{i\theta})$ converges and $K_\nu(xe^{i\theta})$ tends to infinity as $x\downarrow 0$. This yields that

$$\frac{\xi_\nu(c\varepsilon e^{i\theta})}{\xi_\nu(\varepsilon e^{i\theta})}=\frac{K_\nu(c\varepsilon e^{i\theta})+O(1)}{K_\nu(\varepsilon e^{i\theta})+O(1)}=\frac{1}{c^\nu}\{1+o(1)\}$$

as $\varepsilon\downarrow 0$.

We can show

$$\frac{\zeta_\nu(c\varepsilon e^{i\theta})}{\zeta_\nu(\varepsilon e^{i\theta})}=\frac{1}{c^\nu}\{1+o(1)\}$$

in the same fashion. □

The first three terms of the right hand side of (5.12) can be calculated easily. Indeed, Lemma 5.1 yields that the first and the second terms converge to $1/4c^\nu$ and that the third term converges to $1/2c^\nu$. By (4.2) and (5.13), we can easily see

$$\frac{L_{\nu,c}(x)}{\cos(\pi\nu)}=\begin{cases}\dfrac{\log c}{(\log x)^2}\{1+o(1)\} & \text{if } \nu=0,\\[2mm] \dfrac{c^\nu(1-c^{-2\nu})x^{2\nu}}{2^{2\nu-1}\Gamma(\nu)\Gamma(\nu+1)}\{1+o(1)\} & \text{if } \nu>0\end{cases}\tag{5.14}$$

as $x\downarrow 0$, which has been noted in [3], p.29. Hence the last term of the right hand side of (5.12) converges as $\varepsilon\downarrow 0$. Therefore we can conclude

$$\lim_{\varepsilon\downarrow 0}\lim_{R\to\infty}\Pi(R,\varepsilon)=\frac{1}{c^\nu}-\int_0^\infty\frac{we^{-(c-1)x}L_{\nu,c}(x)}{x(x+w)}dx.$$

Since $K_\mu=K_{-\mu}$ for $\mu\geqq 0$, we have that $K_{\nu+1}(z)=K_{|\nu|+1}(z)$ if z is a zero of K_ν. Moreover, we can regard $z_{-\nu,j}$ as $z_{\nu,j}$ for $1\leqq j\leqq N(\nu)$. Therefore we have proven the following.

Theorem 5.2. *Let $c>1$, $\nu\in\mathbb{R}$ and w is a point in D with $K_\nu(w)\neq 0$.*
(1) *If $\nu=\pm 1/2$, we have*
$$\frac{K_\nu(cw)}{K_\nu(w)}=\frac{e^{-(c-1)w}}{c^{|\nu|}}.$$
(2) *If $|\nu|<3/2$ and $\nu\neq\pm 1/2$, we have*
$$\frac{K_\nu(cw)}{K_\nu(w)}=\frac{e^{-(c-1)w}}{c^{|\nu|}}-e^{-(c-1)w}\int_0^\infty\frac{we^{-(c-1)x}L_{|\nu|,c}(x)}{x(x+w)}dx.$$
(3) *If $\nu-1/2$ is an integer and $\nu\neq\pm 1/2$,*
$$\frac{K_\nu(cw)}{K_\nu(w)}=\frac{e^{-(c-1)w}}{c^{|\nu|}}-e^{-(c-1)w}\sum_{j=1}^{N(\nu)}\frac{we^{(c-1)z_{\nu,j}}}{z_{\nu,j}(w-z_{\nu,j})}\frac{K_\nu(cz_{\nu,j})}{K_{\nu+1}(z_{\nu,j})}.$$
(4) *If $\nu-1/2$ is not an integer and $|\nu|>3/2$,*
$$\begin{aligned}\frac{K_\nu(cw)}{K_\nu(w)}=\frac{e^{-(c-1)w}}{c^{|\nu|}}&-e^{-(c-1)w}\sum_{j=1}^{N(\nu)}\frac{we^{(c-1)z_{\nu,j}}}{z_{\nu,j}(w-z_{\nu,j})}\frac{K_\nu(cz_{\nu,j})}{K_{\nu+1}(z_{\nu,j})}\\&-e^{-(c-1)w}\int_0^\infty\frac{we^{-(c-1)x}L_{|\nu|,c}(x)}{x(x+w)}dx.\end{aligned}$$

We are ready to complete our proof of Theorem 2.2. We have
$$\mathcal{L}[G^{(\nu)}_{a,b}](\lambda)=\frac{1}{\alpha^\nu}\frac{K_\nu(\alpha\sqrt{\lambda})}{\lambda K_\nu(\sqrt{\lambda})},\quad \alpha=\frac{a}{b}>1.$$
We need to invert the Laplace transforms of the following functions:
$$\begin{aligned}p_1(\lambda)&=\frac{1}{\lambda}e^{-(\alpha-1)\sqrt{\lambda}},\\ p_2(\lambda;z)&=\frac{1}{\sqrt{\lambda}(\sqrt{\lambda}-z)}e^{-(\alpha-1)\sqrt{\lambda}},\quad z\in\mathbb{C}.\end{aligned}$$
The results may be well known (cf. [13]), but we deduce them from the formula
$$\int_0^\infty e^{-\frac{x^2}{u^2}-\frac{v^2}{x^2}}dx=\frac{u\sqrt{\pi}}{2}e^{-\frac{2v}{u}},\quad u,v>0. \tag{5.15}$$

At first, put
$$q_1(t)=\frac{1}{2\sqrt{\pi t^3}}\int_{\alpha-1}^\infty\xi\{\xi-(\alpha-1)\}e^{-\frac{\xi^2}{4t}}d\xi=\frac{1}{\sqrt{\pi t}}\int_{\alpha-1}^\infty e^{-\frac{\xi^2}{4t}}d\xi.$$
Then we get by (5.15)
$$\begin{aligned}\int_0^\infty e^{-\lambda t}q_1(t)dt&=\frac{1}{2\sqrt{\pi}}\int_{\alpha-1}^\infty\xi\{\xi-(\alpha-1)\}\left[\int_0^\infty e^{-\lambda t-\frac{\xi^2}{4t}}t^{-\frac{3}{2}}dt\right]d\xi\\&=\frac{1}{\sqrt{\pi}}\int_{\alpha-1}^\infty\xi\{\xi-(\alpha-1)\}\left[\int_0^\infty e^{-\frac{\xi^2x^2}{4}-\frac{\lambda}{x^2}}dx\right]d\xi\\&=\int_{\alpha-1}^\infty\{\xi-(\alpha-1)\}e^{-\sqrt{\lambda}\xi}d\xi\\&=\frac{1}{\lambda}e^{-(\alpha-1)\sqrt{\lambda}}\end{aligned}$$

and

$$\mathcal{L}^{-1}[p_1](t) = q_1(t).$$

Next we put

$$q_2(t;z) = \frac{1}{\sqrt{\pi t}} \int_{\alpha-1}^{\infty} e^{-\frac{\xi^2}{4t}+z\{\xi-(\alpha-1)\}} d\xi.$$

Then we obtain from (5.15)

$$\begin{aligned}
\int_0^\infty e^{-\lambda t} q_2(t;z)dt &= \frac{1}{\sqrt{\pi}} \int_{\alpha-1}^{\infty} e^{z\{\xi-(\alpha-1)\}} \left[\int_0^\infty e^{-\lambda t - \frac{\xi^2}{4t}} t^{-\frac{1}{2}} dt \right] d\xi \\
&= \frac{2}{\sqrt{\pi}} \int_{\alpha-1}^{\infty} e^{z\{\xi-(\alpha-1)\}} \left[\int_0^\infty e^{-\lambda x^2 - \frac{\xi^2}{4x^2}} dx \right] d\xi \\
&= \frac{1}{\sqrt{\lambda}} \int_{\alpha-1}^{\infty} e^{z\{\xi-(\alpha-1)\} - \sqrt{\lambda}\xi} d\xi \\
&= \frac{1}{\sqrt{\lambda}(\sqrt{\lambda}-z)} e^{-(\alpha-1)\sqrt{\lambda}}.
\end{aligned}$$

Hence we get

$$\mathcal{L}^{-1}[p_2](t) = q_2(t;z).$$

Now we have shown, for example, for the fourth case where $\nu - 1/2$ is not an integer and $|\nu| > 3/2$,

$$\begin{aligned}
G_{a,b}^{(\nu)}(t) =& \frac{1}{\alpha^{\nu+|\nu|}} \frac{1}{\sqrt{\pi t}} \int_{\alpha-1}^{\infty} e^{-\frac{\xi^2}{4t}} d\xi \\
&- \frac{1}{\alpha^\nu} \frac{1}{\sqrt{\pi t}} \sum_{j=1}^{N(\nu)} \frac{K_\nu(\alpha z_{\nu,j})}{z_{\nu,j} K_{\nu+1}(z_{\nu,j})} \int_{\alpha-1}^{\infty} e^{-\frac{\xi^2}{4t} + z_{\nu,j}\xi} d\xi \\
&- \frac{1}{\alpha^\nu} \frac{1}{\sqrt{\pi t}} \int_{\alpha-1}^{\infty} e^{-\frac{\xi^2}{4t}} \left[\int_0^\infty \frac{L_{|\nu|,\alpha}(x)}{x} e^{-x\xi} dx \right] d\xi.
\end{aligned}$$

Finally, a simple change of variables from ξ to s given by $\xi = (a-b)\sqrt{2t/s}$ gives us the formula in Theorem 2.2 (4). The other cases are simpler.

6. The tail probability of the first hitting time.

As an application of Theorem 2.2, we show the asymptotic behavior of $P(\tau_{a,b}^{(\nu)} > t)$ as $t \to \infty$ when $0 < b < a$. In Section 2 we showed it when $\nu = 0$ by the Tauberian theorem. In [3], it is shown that, if $\nu < 0$,

$$P(\tau_{a,b}^{(\nu)} > t) = c_\nu t^\nu \{1 + o(1)\}$$

holds for some constant c_ν. It should also be noted that, in [16], Yamazato has discussed on the tail probability in a general framework, and some Bessel processes may be treated. We give an explicit expression for the constant c_ν.

To make the statement clear, we define two constants when $\nu - 1/2$ is an integer. Put

$$\sigma_1^{(\nu)} = \frac{(a-b)^{2|\nu|}}{2|\nu|}.$$

Moreover we set $\sigma_2^{(\nu)} = 0$ if $|\nu| = \pm 1/2$ and

$$\sigma_2^{(\nu)} = b^{2|\nu|}(2|\nu|-1)! \sum_{j=1}^{N(\nu)} \frac{K_\nu(az_{\nu,j}/b)}{z_{\nu,j}^{2|\nu|+1} K_{\nu+1}(z_{\nu,j})} e^{\frac{z_{\nu,j}(a-b)}{b}} \sum_{k=0}^{2|\nu|-1} \frac{1}{k!} \Big\{ -\frac{z_{\nu,j}(a-b)}{b} \Big\}^k,$$

if otherwise.

Theorem 6.1. *Let $0 < b < a$.*
(1) *If $\nu = 0$,*

$$P(\tau_{a,b}^{(0)} > t) = \frac{2\log(a/b)}{\log t} + o\Big(\frac{1}{\log t}\Big).$$

(2) *If $\nu > 0$ and $\nu - 1/2$ is an integer*

$$P(\tau_{a,b}^{(\nu)} > t) = 1 - \Big(\frac{b}{a}\Big)^{2\nu} + \Big(\frac{b}{a}\Big)^{2\nu} \sqrt{\frac{2}{\pi}} \frac{(-1/2)^{\nu-1/2}}{(\nu-1/2)!} \Big\{ \sigma_1^{(\nu)} + \Big(\frac{a}{b}\Big)^\nu \sigma_2^{(\nu)} \Big\} \frac{1}{t^\nu} + O\Big(\frac{1}{t^{\nu+1}}\Big).$$

(3) *If $\nu < 0$ and $\nu - 1/2$ is an integer*

$$P(\tau_{a,b}^{(\nu)} > t) = \sqrt{\frac{2}{\pi}} \frac{(-1/2)^{-\nu-1/2}}{(-\nu-1/2)!} \Big\{ \sigma_1^{(\nu)} + \Big(\frac{b}{a}\Big)^\nu \sigma_2^{(\nu)} \Big\} t^\nu + O(t^{\nu-1}).$$

(4) *If $\nu > 0$ and $\nu - 1/2$ is not an integer,*

$$P(\tau_{a,b}^{(\nu)} > t) = 1 - \Big(\frac{b}{a}\Big)^{2\nu} + \Big(\frac{b^3}{2a}\Big)^\nu \Big\{ \Big(\frac{a}{b}\Big)^\nu - \Big(\frac{b}{a}\Big)^\nu \Big\} \frac{1}{\Gamma(1+\nu)t^\nu} + o\Big(\frac{1}{t^\nu}\Big).$$

(5) *If $\nu < 0$ and $\nu - 1/2$ is not an integer,*

$$P(\tau_{a,b}^{(\nu)} > t) = \Big(\frac{2}{ab}\Big)^\nu \Big\{ \Big(\frac{b}{a}\Big)^\nu - \Big(\frac{a}{b}\Big)^\nu \Big\} \frac{t^\nu}{\Gamma(1-\nu)} + o(t^\nu).$$

Remark 6.2. It seems that (4) and (5) also hold when $\nu - 1/2$. But we do not pursue the identities.

Before proving this theorem, we give two lemmas. The following one is the immediate consequence of Lemma 4.3 given in [3]. We let $m(\nu)$ be the greatest integer which is not larger than $|\nu| - 1/2$.

Lemma 6.3. *We assume that $|\nu| \geqq 1/2$. If $\nu - 1/2$ is an integer, we have that, for any $0 \leqq m \leqq m(\nu) - 1$*

$$\lim_{t\to\infty} t^{m+1/2} P(\tau_{a,b}^{(\nu)} > t) = 0. \tag{6.1}$$

If $\nu - 1/2$ is not an integer, we have (6.1) for any $0 \leqq m \leqq m(\nu)$.

For $t > 0$, $\nu \neq 0$ and $z \in \mathbb{C}^-$ set

$$\Psi_1(t) = 1 - \int_0^t \frac{a-b}{\sqrt{2\pi s^3}} e^{-\frac{(a-b)^2}{2s}} ds,$$

$$\Psi_2(t;z) = \int_0^t \frac{a-b}{\sqrt{2\pi s^3}} e^{-\frac{(a-b)^2}{2s} + \frac{z(a-b)\sqrt{t}}{b\sqrt{s}}} ds,$$

$$\Psi_3(t;\nu) = \int_0^t \frac{a-b}{\sqrt{2\pi s^3}} e^{-\frac{(a-b)^2}{2s}} \left[\int_0^\infty \frac{L_{|\nu|,a/b}(x)}{x} e^{-\frac{x(a-b)\sqrt{t}}{b\sqrt{s}}} dx \right] ds.$$

Theorem 2.2 implies that $P(\tau_{a,b}^{(\nu)} > t)$ is represented by a linear combination of Ψ_i's. Changing variables from s to u given by $(a-b)/\sqrt{s} = u$, we have

$$\Psi_1(t) = 1 - \sqrt{\frac{2}{\pi}} \int_{(a-b)/\sqrt{t}}^\infty e^{-\frac{u^2}{2}} du = \sqrt{\frac{2}{\pi}} \int_0^{(a-b)/\sqrt{t}} e^{-\frac{u^2}{2}} du,$$

$$\Psi_2(t;z) = \sqrt{\frac{2}{\pi}} \int_{(a-b)/\sqrt{t}}^\infty e^{-\frac{u^2}{2} + \frac{z\sqrt{t}u}{b}} du,$$

$$\Psi_3(t;\nu) = \sqrt{\frac{2}{\pi}} \cos(\pi\nu) \Psi_3^0(t;\nu),$$

where

$$\Psi_3^0(t;\nu) = \int_{(a-b)/\sqrt{t}}^\infty e^{-\frac{u^2}{2}} \left[\int_0^\infty \frac{L_{|\nu|,a/b}^0(x)}{x} e^{-\frac{x\sqrt{t}u}{b}} dx \right] du$$

and $L_{|\nu|,a/b}^0(x) = L_{|\nu|,a/b}(x)/\cos(\pi\nu)$. It is obvious that $L_{|\nu|,a/b}^0$ is positive on $(0,\infty)$ since I_ν and K_ν is increasing and decreasing on $(0,\infty)$, respectively. For an integer m with $0 \leqq m \leqq m(\nu)$, we set

$$\beta_1(m) = \frac{(a-b)^{2m+1}}{2m+1},$$

$$\beta_2^{(\nu)}(m) = \sum_{j=1}^{N(\nu)} \frac{K_\nu(az_{\nu,j}/b)}{z_{\nu,j} K_{\nu+1}(z_{\nu,j})} \beta(m; z_{\nu,j}),$$

$$\beta_3^{(\nu)}(m) = (2m)!\, b^{2m+1} \sum_{k=0}^{2m} \frac{1}{k!} \left(\frac{a-b}{b} \right)^k \int_0^\infty \frac{L_{|\nu|,a/b}(x)}{x^{2m-k+2}} e^{-\frac{x(a-b)}{b}} dx.$$

where

$$\beta(m;z) = -(2m)! \left(\frac{b}{z} \right)^{2m+1} e^{\frac{z(a-b)}{b}} \sum_{k=0}^{2m} \frac{1}{k!} \left\{ -\frac{z(a-b)}{b} \right\}^k.$$

It follows from (3.7) and (5.10) that

$$L_{|\nu|,a/b}^0(x) = \pi \sqrt{\frac{b}{a}} e^{(a/b-3)x} \{1 + o(1)\}$$

as $x \to \infty$. Moreover, (5.14) yields that $L_{|\nu|,a/b}^0(x)/x^{2m-k+2}$ is asymptotically equal to a constant multiple of $1/x^{2m-k+2-2|\nu|}$ for small x. Since

$$2m - k + 2 - 2|\nu| \leqq 2m(\nu) + 2 - 2|\nu| < 1,$$

we have that the following improper integral converges:

$$\int_0^\infty \frac{L_{|\nu|,a/b}(x)}{x^{2m-k+2}} e^{-\frac{x(a-b)}{b}} dx = \cos(\pi\nu) \int_0^\infty \frac{L_{|\nu|,a/b}^0(x)}{x^{2m-k+2}} e^{-\frac{x(a-b)}{b}} dx.$$

Lemma 6.4. *If* $|\nu| \geqq 1/2$,

$$\Psi_1(t) = \sqrt{\frac{2}{\pi}} \sum_{m=0}^{m(\nu)} \frac{(-1/2)^m \beta_1(m)}{m!} \frac{1}{t^{m+1/2}} + O\Big(\frac{1}{t^{m(\nu)+3/2}}\Big), \tag{6.2}$$

$$\Psi_2(t;z) = \sqrt{\frac{2}{\pi}} \sum_{m=0}^{m(\nu)} \frac{(-1/2)^m \beta(m;z)}{m!} \frac{1}{t^{m+1/2}} + O\Big(\frac{1}{t^{m(\nu)+3/2}}\Big). \tag{6.3}$$

If $0 < |\nu| < 1/2$,

$$\Psi_3(t;\nu) = \Big(\frac{b^2}{2}\Big)^{|\nu|} \Big\{ \Big(\frac{a}{b}\Big)^{|\nu|} - \Big(\frac{b}{a}\Big)^{|\nu|} \Big\} \frac{1}{\Gamma(1+|\nu|) t^{|\nu|}} + o\Big(\frac{1}{t^{|\nu|}}\Big). \tag{6.4}$$

If $|\nu| > 1/2$ *and* $\nu - 1/2$ *is not an integer,*

$$\begin{aligned} \Psi_3(t;\nu) =& \sqrt{\frac{2}{\pi}} \sum_{m=0}^{m(\nu)} \frac{(-1/2)^m \beta_3^{(\nu)}(m)}{m!} \frac{1}{t^{m+1/2}} \\ &+ \Big(\frac{b^2}{2}\Big)^{|\nu|} \Big\{ \Big(\frac{a}{b}\Big)^{|\nu|} - \Big(\frac{b}{a}\Big)^{|\nu|} \Big\} \frac{1}{\Gamma(1+|\nu|) t^{|\nu|}} \\ &+ o\Big(\frac{1}{t^{|\nu|}}\Big). \end{aligned} \tag{6.5}$$

Proof. For $x \geqq 0$ let

$$P^{(\nu)}(x) = e^{-x^2/2} - \sum_{m=0}^{m(\nu)} \frac{1}{m!} \Big(-\frac{x^2}{2}\Big).$$

Note that

$$|P^{(\nu)}(x)| \leqq \frac{x^{2m(\nu)+2}}{2^{m(\nu)+1}\{m(\nu)+1\}!}. \tag{6.6}$$

Hence we have

$$\Psi_1(t) = \sqrt{\frac{2}{\pi}} \sum_{m=0}^{m(\nu)} \frac{1}{m!} \Big(-\frac{1}{2}\Big)^m \int_0^{(a-b)/\sqrt{t}} u^{2m} du + \sqrt{\frac{2}{\pi}} \int_0^{(a-b)/\sqrt{t}} P^{(\nu)}(u) du,$$

which implies (6.2). Similarly, by the formula

$$\int_\beta^\infty x^n e^{-\mu x} dx = e^{-\beta\mu} \sum_{k=0}^{n} \frac{n!}{k!} \frac{\beta^k}{\mu^{n-k+1}} \tag{6.7}$$

for $\beta > 0$ and $\mathrm{Re}(\mu) > 0$ (cf. [7], p.340), we immediately get (6.3).

For $|\nu| > 1/2$ and $\nu - 1/2$ is not an integer we have

$$\begin{aligned}\Psi_3^0(t;\nu) &= \sum_{m=0}^{m(\nu)} \frac{1}{m!}\Big(-\frac{1}{2}\Big)^m \int_{(a-b)/\sqrt{t}}^\infty u^{2m}\Big[\int_0^\infty \frac{L^0_{|\nu|,a/b}(x)}{x} e^{-\frac{x\sqrt{t}u}{b}}\,dx\Big]du \\ &\quad + \int_{(a-b)/\sqrt{t}}^\infty P^{(\nu)}(u)\Big[\int_0^\infty \frac{L^0_{|\nu|,a/b}(x)}{x} e^{-\frac{x\sqrt{t}u}{b}}\,dx\Big]du.\end{aligned}$$

Then the first term of the right hand side is equal to

$$\sum_{m=0}^{m(\nu)} \frac{(-1/2)^m \beta_3^{(\nu)}(m)}{\cos(\pi\nu)m!}\frac{1}{t^{m+1/2}}$$

since

$$\int_{(a-b)/\sqrt{t}}^\infty u^{2m}\Big[\int_0^\infty \frac{L^0_{|\nu|,a/b}(x)}{x} e^{-\frac{x\sqrt{t}u}{b}}\,dx\Big]du = \frac{\beta_3^{(\nu)}(m)}{t^{m+1/2}},$$

which is obtained by the Fubini theorem and (6.7). We set

$$\widetilde{\Psi}_3^0(t;\nu) = \int_{(a-b)/\sqrt{t}}^\infty P^{(\nu)}(u)\Big[\int_0^\infty \frac{L^0_{|\nu|,a/b}(x)}{x} e^{-\frac{x\sqrt{t}u}{b}}\,dx\Big]du.$$

Changing variables from x to y given by $x\sqrt{t}u/b = y$, we have

$$\widetilde{\Psi}_3^0(t;\nu) = \int_0^\infty 1_{\left[\frac{a-b}{\sqrt{t}},\infty\right)}(u)P^{(\nu)}(u)\Big[\int_0^\infty L^0_{|\nu|,a/b}\Big(\frac{by}{\sqrt{t}u}\Big)\frac{e^{-y}}{y}dy\Big]du,$$

where 1_A is the indicator function of A. To see the convergence of $t^{|\nu|}\widetilde{\Psi}_3^0(t;\nu)$ as $t \to \infty$, we need to dominate

$$t^{|\nu|}1_{\left[\frac{a-b}{\sqrt{t}},\infty\right)}(u)|P^{(\nu)}(u)|L^0_{|\nu|,a/b}\Big(\frac{by}{\sqrt{t}u}\Big)\frac{e^{-y}}{y} \tag{6.8}$$

by an integrable function which is independent of t. We have that (6.8) is equal to

$$b^{2|\nu|}1_{\left[\frac{a-b}{\sqrt{t}},\infty\right)}(u)|P^{(\nu)}(u)|\frac{L^0_{|\nu|,a/b}(by/\sqrt{t}u)}{(by/\sqrt{t}u)^{2|\nu|}}y^{2|\nu|-1}u^{-2|\nu|}e^{-y}. \tag{6.9}$$

Since

$$\frac{L^0_{|\nu|,a/b}(x)}{x^{2|\nu|}}e^{-(a/b-3)x}$$

is bounded on $(0,\infty)$, we have that (6.9) is dominated by a constant multiple of

$$1_{\left[\frac{a-b}{\sqrt{t}},\infty\right)}(u)|P^{(\nu)}(u)|e^{\frac{y(a-3b)}{\sqrt{t}u}-y}y^{2|\nu|-1}u^{-2|\nu|}. \tag{6.10}$$

We have that, if $a \leqq 3b$,

$$e^{\frac{y(a-3b)}{\sqrt{t}u}-y} \leqq e^{-y}$$

and that, if $a > 3b$,
$$e^{\frac{y(a-3b)}{\sqrt{t}u}-y} \leqq e^{\frac{y(a-3b)}{a-b}-y} = e^{-\frac{2b}{a-b}y}$$
for $u \geqq (a-b)/\sqrt{t}$. Let
$$\kappa = \min\left\{1, \frac{2b}{a-b}\right\}$$
and hence (6.10) is bounded by
$$|P^{(\nu)}(u)|u^{-2|\nu|}y^{2|\nu|-1}e^{-\kappa y}. \tag{6.11}$$
To see that $|P^{(\nu)}(u)|u^{-2|\nu|}$ is integrable on $(0,\infty)$, we note that
$$|P^{(\nu)}(x)| \leqq C_{14} \min\{1, x^2/2\}x^{2m(\nu)}$$
for some constant C_{14}. Then we get
$$|P^{(\nu)}(x)|u^{-2|\nu|} \leqq \begin{cases} C_{14}u^{2m(\nu)-2|\nu|+2} & \text{if } 0 < u \leqq \sqrt{2}, \\ C_{14}u^{2m(\nu)-2|\nu|} & \text{if } u > \sqrt{2}. \end{cases}$$
Since
$$\begin{aligned} &2m(\nu) - 2|\nu| + 2 > 2\left(|\nu| - \frac{3}{2}\right) - 2|\nu| + 2 > -1, \\ &2m(\nu) - 2|\nu| < 2\left(|\nu| - \frac{1}{2}\right) - 2|\nu| < -1, \end{aligned}$$
we see that the function given by (6.11) is integrable on $(0,\infty)\times(0,\infty)$. Applying the dominated convergence theorem, the Fubini theorem and (5.14), we have that $t^{|\nu|}\widetilde{\Psi}_3^0(t;\nu)$ tends to
$$\frac{b^{2|\nu|}(a/b)^{|\nu|}\{1-(a/b)^{-2|\nu|}\}}{2^{2|\nu|-1}\Gamma(|\nu|)\Gamma(|\nu|+1)} \int_0^\infty P^{(\nu)}(u)u^{-2|\nu|}du \int_0^\infty e^{-y}y^{2|\nu|-1}dy \tag{6.12}$$
as $t \to \infty$. Since
$$\int_0^\infty \frac{e^{-y}}{y^{1-2|\nu|}}dy = \frac{2^{2|\nu|-1}}{\sqrt{\pi}}\Gamma(|\nu|)\Gamma\left(\frac{1}{2}+|\nu|\right)$$
(cf. [12], p.3), (6.12) coincides with
$$\frac{b^{2|\nu|}}{\sqrt{\pi}}\left\{\left(\frac{a}{b}\right)^{|\nu|} - \left(\frac{a}{b}\right)^{-|\nu|}\right\}\frac{1}{\Gamma(1+|\nu|)}\Gamma\left(\frac{1}{2}+|\nu|\right)\int_0^\infty P^{(\nu)}(u)u^{-2|\nu|}du.$$
Changing variables from u to v given by $v = u^2/2$, we have
$$\int_0^\infty P^{(\nu)}(u)u^{-2|\nu|}du = \frac{1}{2^{|\nu|+1/2}}\int_0^\infty \frac{1}{v^{|\nu|+1/2}}\left\{e^{-v} - \sum_{m=0}^{m(\nu)}\frac{(-1)^m}{m!}v^m\right\}dv,$$

which is equal to

$$\frac{1}{2^{|\nu|+1/2}}\Gamma\left(\frac{1}{2}-|\nu|\right)$$

(cf. [7], p.361). The formula

$$\Gamma\left(\frac{1}{2}+|\nu|\right)\Gamma\left(\frac{1}{2}-|\nu|\right)=\frac{\pi}{\cos(\pi\nu)}$$

(cf. [12], p.3) immediately yields

$$\widetilde{\Psi}_3^0(t;\nu)=\sqrt{\frac{\pi}{2}}\frac{1}{\cos(\pi\nu)}\left(\frac{b^2}{2}\right)^{|\nu|}\left\{\left(\frac{a}{b}\right)^{|\nu|}-\left(\frac{b}{a}\right)^{|\nu|}\right\}\frac{1}{\Gamma(1+|\nu|)t^{|\nu|}}+o\left(\frac{1}{t^{|\nu|}}\right)$$

and hence, we have (6.5).

When $0<|\nu|<1/2$, it is enough to consider $\Psi_3^0(t;\nu)$ directly. We can easily deduce (6.4) in the same way as $\widetilde{\Psi}_3^0(t;\nu)$ for $|\nu|>1/2$. The calculation is left to the reader. □

We are now ready to prove Theorem 6.1. We need only to show Theorem 6.1 in the case of $\nu\neq 0$.

For simplicity we set $c=b/a$ and

$$w_j^{(\nu)}=\frac{K_\nu(az_{\nu,j}/b)}{z_{\nu,j}K_{\nu+1}(z_{\nu,j})}$$

for $1\leqq j\leqq N(\nu)$. We first consider the case when $\nu-1/2$ is not an integer and $\nu<0$. In this case, if $\nu<-3/2$, Theorem 2.2 gives

$$P(\tau_{a,b}^{(\nu)}>t)=\Psi_1(t)+c^\nu\sum_{j=1}^{N(\nu)}w_j^{(\nu)}\Psi_2(t;z_{\nu,j})+c^\nu\Psi_3(t;\nu).$$

Note that $m(\nu)+1/2<|\nu|<m(\nu)+3/2$. It follows from Lemma 6.4 that

$$\begin{aligned}P(\tau_{a,b}^{(\nu)}>t)=&\sqrt{\frac{2}{\pi}}\sum_{m=0}^{m(\nu)}\frac{(-1/2)^m}{m!\,t^{m+1/2}}\{\beta_1(m)+c^\nu\beta_2^{(\nu)}(m)+c^\nu\beta_3^{(\nu)}(m)\}\\&+c^\nu\left(\frac{b^2}{2}\right)^{-\nu}(c^\nu-c^{-\nu})\frac{t^\nu}{\Gamma(1-\nu)}+o(t^\nu).\end{aligned}$$

By virtue of Lemma 6.3, we have that, for any $0\leqq m\leqq m(\nu)$

$$\beta_1(m)+c^\nu\beta_2^{(\nu)}(m)+c^\nu\beta_3^{(\nu)}(m)=0. \tag{6.13}$$

This immediately yields

$$P(\tau_{a,b}^{(\nu)}>t)=\left(\frac{2}{ab}\right)^\nu(c^\nu-c^{-\nu})\frac{t^\nu}{\Gamma(1-\nu)}+o(t^\nu). \tag{6.14}$$

If $-3/2 < \nu < 0$, Theorem 2.2 gives

$$P(\tau_{a,b}^{(\nu)} > t) = \Psi_1(t) + c^\nu \Psi_3(t;\nu).$$

In the case of $-3/2 < \nu < -1/2$, we have by Lemma 6.4 that

$$P(\tau_{a,b}^{(\nu)} > t) = \sqrt{\frac{2}{\pi t}}\{\beta_1(0) + c^\nu \beta_3^{(\nu)}(0)\} + \left(\frac{2}{ab}\right)^\nu (c^\nu - c^{-\nu})\frac{t^\nu}{\Gamma(1-\nu)} + o(t^\nu).$$

Lemma 6.3 yields

$$\beta_1(0) + c^\nu \beta_3^{(\nu)}(0) = 0,$$

and hence we have (6.14). In the case of $-1/2 < \nu < 0$, Lemma 6.4 gives (6.14) since $\Psi_1(t)$ is of order $1/\sqrt{t}$. We therefore obtain Theorem 6.1 (5).

We next consider the case when $\nu - 1/2$ is not an integer and $\nu > 0$. If $\nu > 3/2$, it follows from Theorem 2.2 that

$$P(\tau_{a,b}^{(\nu)} > t) = 1 - c^{2\nu} + c^{2\nu}\Psi_1(t) + c^\nu \sum_{j=1}^{N(\nu)} w_j^{(\nu)} \Psi_2(t; z_{\nu,j}) + c^\nu \Psi_3(t;\nu),$$

which is equal to

$$\begin{aligned} 1 - c^{2\nu} + \sqrt{\frac{2}{\pi}} \sum_{m=0}^{m(\nu)} \frac{(-1/2)^m c^{2\nu}}{m!\, t^{m+1/2}} \{\beta_1(m) + c^{-\nu}\beta_2^{(\nu)}(m) + c^{-\nu}\beta_3^{(\nu)}(m)\} \\ + c^\nu \left(\frac{b^2}{2}\right)^\nu (c^{-\nu} - c^\nu)\frac{1}{\Gamma(1+\nu)t^\nu} + O\left(\frac{1}{t^\nu}\right). \end{aligned}$$

We have that (6.13) is equivalent to

$$\beta_1(m) + c^{-\nu}\beta_2^{(-\nu)}(m) + c^{-\nu}\beta_3^{(-\nu)}(m) = 0$$

for $0 \leqq m \leqq m(\nu)$. Since $z_{-\nu,j}$ is regarded as $z_{\nu,j}$ and $K_{-\nu+1}(z_{-\nu,j}) = K_{\nu+1}(z_{\nu,j})$ for $1 \leqq j \leqq N(\nu)$, we obtain that $\beta_2^{(\nu)}(m) = \beta_2^{(-\nu)}(m)$. Moreover it is trivial that $\beta_3^{(\nu)}(m) = \beta_3^{(-\nu)}(m)$, and hence we have that, for $0 \leqq m \leqq m(\nu)$

$$\beta_1(m) + c^{-\nu}\beta_2^{(\nu)}(m) + c^{-\nu}\beta_3^{(\nu)}(m) = 0.$$

This immediately yields

$$P(\tau_{a,b}^{(\nu)} > t) = 1 - c^{2\nu} + \left(\frac{b^3}{2a}\right)^\nu (c^{-\nu} - c^\nu)\frac{1}{\Gamma(1+\nu)t^\nu} + O\left(\frac{1}{t^\nu}\right). \tag{6.15}$$

In the other cases, (6.15) can be derived in a similar way. Then we finish to prove Theorem 6.1 (4).

We lastly consider the case when $\nu - 1/2$ is an integer. Similarly to the case when $\nu - 1/2$ is not an integer, we can deduce the asymptotic behavior for $\nu > 0$

from that for $\nu < 0$. Hence we shall treat only the case of $\nu < 0$. If $\nu \neq -1/2$, Theorem 2.2 and Lemma 6.4 give

$$\begin{aligned} P(\tau_{a,b}^{(\nu)} > t) &= \Psi_1(t) + c^\nu \sum_{j=1}^{N(\nu)} w_j^{(\nu)} \Psi_2(t; z_{\nu,j}) \\ &= \sqrt{\frac{2}{\pi}} \sum_{m=0}^{m(\nu)} \frac{(-1/2)^m}{m!\, t^{m+1/2}} \{\beta_1(m) + c^\nu \beta_2^{(\nu)}(m)\} + O\Big(\frac{1}{t^{m(\nu)+3/2}}\Big). \end{aligned}$$

Note that $m(\nu) = -\nu - 1/2 \geqq 1$. It follows from Lemma 6.3 that

$$\begin{aligned} P(\tau_{a,b}^{(\nu)} > t) &= \sqrt{\frac{2}{\pi}} \frac{(-1/2)^{m(\nu)}}{\{m(\nu)\}!\, t^{m(\nu)+1/2}} \{\beta_1(m(\nu)) + c^\nu \beta_2^{(\nu)}(m(\nu))\} + O(t^{\nu-1}) \\ &= \sqrt{\frac{2}{\pi}} \frac{(-1/2)^{-\nu-1/2}}{(-\nu-1/2)!} \{\sigma_1^{(\nu)} + c^\nu \sigma_2^{(\nu)}\} t^\nu + O(t^{\nu-1}). \end{aligned}$$

Since

$$P(\tau_{a,b}^{(\nu)} > t) = \Psi_1(t) = \sqrt{\frac{2}{\pi t}}(a-b) + O\Big(\frac{1}{t^{3/2}}\Big)$$

if $\nu = -1/2$, we obtain Theorem 6.1 (3). We finish to prove Theorem 6.1.

7. An addition formula for the sum of Bessel functions ratio

From the results in Section 4, we have that, if $\nu > -1$,

$$\begin{aligned} E[e^{-\lambda \tau_{a,b}^{(\nu)}}] &= 1 - 4b^2 \Big(\frac{b}{a}\Big)^\nu \sum_{k=1}^\infty \frac{\lambda}{j_{\nu,k}(2b^2\lambda + j_{\nu,k}^2)} \frac{J_\nu(a j_{\nu,k}/b)}{J_{\nu+1}(j_{\nu,k})} \\ &= 1 - 2\Big(\frac{b}{a}\Big)^\nu \sum_{k=1}^\infty \Big(\frac{1}{j_{\nu,k}} - \frac{j_{\nu,k}}{2b^2\lambda + j_{\nu,k}^2}\Big) \frac{J_\nu(a j_{\nu,k}/b)}{J_{\nu+1}(j_{\nu,k})}. \end{aligned}$$

Hence, heuristically, we obtain the following addition formula, which may be of independent interest. In fact the idea of proof is the same as that in Section 4.

Theorem 7.1. *Let $\mu > -1$ and $0 < c < 1$. Then we have*

$$\sum_{n=1}^\infty \frac{J_\mu(c j_{\mu,n})}{j_{\mu,n} J_{\mu+1}(j_{\mu,n})} = \frac{1}{2} c^\mu.$$

Proof. Letting $\varphi_{\mu,c}$ be the holomorphic function considered in Section 4, we put

$$l_{\mu,c}(z) = \frac{\varphi_{\mu,c}(z)}{z \varphi_{\mu,1}(z)}$$

for $z \in \mathbb{C}$ with $\varphi_{\mu,1}(z) \neq 0$ and

$$\Sigma(R) = \frac{1}{2\pi i} \int_{C(R)} l_{\mu,c}(z) dz$$

for $R>0$. We take n so large that (4.4) holds. Hence the residue theorem yields that

$$\Sigma(R_n) = \mathrm{Res}(0; l_{\mu,c}) + \sum_{k=1}^{n} \mathrm{Res}(j_{\mu,k}e^{i\pi/2}; l_{\mu,c}) + \sum_{k=1}^{n} \mathrm{Res}(j_{\mu,k}e^{-i\pi/2}; l_{\mu,c}).$$

It is easy to see

$$\mathrm{Res}(0; l_{\mu,c}) = c^{\mu}.$$

We can show, in a similar way to (4.5),

$$\mathrm{Res}(j_{\mu,k}e^{i\pi/2}; l_{\mu,c}) = -\frac{J_{\mu}(cj_{\mu,k})}{j_{\mu,k}J_{\mu+1}(j_{\mu,k})}.$$

and, in a similar way to (4.6),

$$\mathrm{Res}(j_{\mu,k}e^{-i\pi/2}; l_{\mu,c}) = -\frac{J_{\mu}(cj_{\mu,k})}{j_{\mu,k}J_{\mu+1}(j_{\mu,k})}.$$

Therefore it follows that

$$\Sigma(R_n) = c^{\mu} - 2\sum_{k=1}^{n} \frac{J_{\mu}(cj_{\mu,k})}{j_{\mu,k}J_{\mu+1}(j_{\mu,k})}.$$

We now prove that $\Sigma(R_n)$ tends to 0 as $n\to\infty$. We have

$$\Sigma(R_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\varphi_{\mu,c}(R_n e^{i\theta})}{\varphi_{\mu,1}(R_n e^{i\theta})} d\theta,$$

which is equal to the sum of the following three integrals:

$$\Sigma_1(R_n) = \frac{1}{2\pi}\int_{-\pi}^{-\pi/2} \frac{\varphi_{\mu,c}(R_n e^{i\theta})}{\varphi_{\mu,1}(R_n e^{i\theta})} d\theta,$$
$$\Sigma_2(R_n) = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \frac{\varphi_{\mu,c}(R_n e^{i\theta})}{\varphi_{\mu,1}(R_n e^{i\theta})} d\theta,$$
$$\Sigma_3(R_n) = \frac{1}{2\pi}\int_{\pi/2}^{\pi} \frac{\varphi_{\mu,c}(R_n e^{i\theta})}{\varphi_{\mu,1}(R_n e^{i\theta})} d\theta.$$

Since $\varphi_{\mu,c}(ze^{\pm i\pi}) = \varphi_{\mu,c}(z)$, the change variable formula yields

$$\Sigma_1(R_n) = \frac{1}{2\pi}\int_{0}^{\pi/2} \frac{\varphi_{\mu,c}(R_n e^{i(\theta-\pi)})}{\varphi_{\mu,1}(R_n e^{i(\theta-\pi)})} d\theta = \frac{1}{2\pi}\int_{0}^{\pi/2} \frac{I_{\mu}(cR_n e^{i\theta})}{I_{\mu}(R_n e^{i\theta})} d\theta$$

and

$$\Sigma_3(R_n) = \frac{1}{2\pi}\int_{-\pi/2}^{0} \frac{\varphi_{\mu,c}(R_n e^{i(\theta+\pi)})}{\varphi_{\mu,1}(R_n e^{i(\theta+\pi)})} d\theta = \frac{1}{2\pi}\int_{-\pi/2}^{0} \frac{I_{\mu}(cR_n e^{i\theta})}{I_{\mu}(R_n e^{i\theta})} d\theta.$$

Hence $\Sigma_2(R_n) = \Sigma_1(R_n) + \Sigma_3(R_n)$ and

$$|\Sigma(R_n)| \leqq \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \left| \frac{I_{\mu}(cR_n e^{i\theta})}{I_{\mu}(R_n e^{i\theta})} \right| d\theta. \tag{7.1}$$

Let $\eta \geqq 1$ be given and set
$$\delta_n = \operatorname{Arcsin} \frac{\eta}{R_n}.$$
Note that δ_n tends to 0 as $n \to \infty$ and that the right hand side of (7.1) is equal to

$$\begin{aligned}\frac{1}{\pi}\int_{-\pi/2+\delta_n}^{\pi/2-\delta_n} \left|\frac{I_\mu(cR_ne^{i\theta})}{I_\mu(R_ne^{i\theta})}\right| d\theta + \frac{1}{\pi}\int_{\pi/2-\delta_n}^{\pi/2} \left|\frac{I_\mu(cR_ne^{i\theta})}{I_\mu(R_ne^{i\theta})}\right| d\theta \\ + \frac{1}{\pi}\int_{-\pi/2}^{-\pi/2+\delta_n} \left|\frac{I_\mu(cR_ne^{i\theta})}{I_\mu(R_ne^{i\theta})}\right| d\theta.\end{aligned} \tag{7.2}$$

For the first term, by using Lemma 3.1, we get
$$\int_{-\pi/2+\delta_n}^{\pi/2-\delta_n} \left|\frac{I_\mu(cR_ne^{i\theta})}{I_\mu(R_ne^{i\theta})}\right| d\theta \leqq C_{15}e^{-2(1-c)\eta}$$
for some constant C_{15}. From (3.1) and (3.2) it is easy to see for the second term of (7.2)
$$\int_{\pi/2-\delta_n}^{\pi/2} \left|\frac{I_\mu(cR_ne^{i\theta})}{I_\mu(R_ne^{i\theta})}\right| d\theta = \int_{-\delta_n}^{0} \left|\frac{I_\mu(cR_ne^{i(\theta+\pi/2)})}{I_\mu(R_ne^{i(\theta+\pi/2)})}\right| d\theta = \int_{-\delta_n}^{0} \left|\frac{J_\mu(cR_ne^{i\theta})}{J_\mu(R_ne^{i\theta})}\right| d\theta$$
and from Lemma 3.2
$$\int_{-\delta_n}^{0} \left|\frac{J_\mu(cR_ne^{i\theta})}{J_\mu(R_ne^{i\theta})}\right| d\theta \leqq C_{16}e^{c\eta}\delta_n.$$
The third term of (7.2) is estimated from above in the same way.

Hence we get
$$\limsup_{n\to\infty} |\Sigma(R_n)| \leqq C_{15}e^{-2(1-c)\eta}$$
for an arbitrary $\eta \geqq 1$ and, therefore, $\Sigma(R_n)$ tends to 0 as $n \to \infty$. □

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